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RESEARCH STATEMENT

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Affine Kac-Moody Lie algebras are one of the most interesting families of infinite-dimensional Lie algebras. There are lots of applications of them in for example, representation theory, mathematical physics, string theory, ring theory, number theory, combinatorics, vertex operator theory, etc. Also, the structure of affine Kac-Moody Lie algebras is as beautiful as that of the finite-dimensional simple Lie algebras. The history of the development of the Kac-Moody Lie algebras from the finite-dimensional Lie algebras is very interesting, too (see [BP]).

My primary interest is in extended affine Lie algebras or EALAs for short. EALAs form a new class of infinite-dimensional Lie algebras, which were first introduced by the physicists Høegh-Krohn and Torresani in 1990 [HT] (under the name of irreducible quasi-simple Lie algebras) as a generalization of the finite-dimensional simple Lie algebras and affine Kac-Moody Lie algebras over the complex numbers. They have been systematically studied in [AABGP]. To an EALA corresponds a finite irreducible root system Δ (possibly nonreduced), called its type. In the classification of EALAs, it is crucial to classify the cores which are certain important ideals. The cores of EALAs of reduced type except $\Delta = \mathrm{A}_1$ were classified in [BGK], [BGKN] and [AG]. The cores of EALAs of type A_1 are coordinatized by Jordan algebras. They are central extensions of so-called Tits-Kantor-Koecher algebras constructed from Jordan algebras.

In my Ph.D thesis I studied the coordinate algebra of EALAs of type A_1 and related topics. It turns out that the coordinate algebra is a certain $\mathbb{Z}^n$-graded Jordan algebra, called a *Jordan n-torus* or simply a *Jordan torus*, which can be considered as a Jordan analogue of the algebra of Laurent polynomials in n variables. Since the Jordan tori turn out to be strongly prime, I would use Zelmanov's Prime Structure Theorem [MZ] as the first step of my classification. Thus, a Jordan torus is either of Hermitian, Clifford or Albert type. For each type we then determine the possible Jordan tori in [Y1]. The invariant bilinear forms and derivations of Jordan tori have been classified in [NY]. Using the result from [ABG1], the authors have classified the cores of EALAs of type BC_r for $r \geq 3$ in [ABG2]. Thus the remaining types are BC_2 and BC_1. I am interested in the type BC_1 since the coordinate algebra is a natural generalization of Jordan tori, called a *structurable n-torus* or simply a *structurable torus*. Structurable algebras are certain nonassociative algebras with involution which were introduced by Allison in 1978 [A] to describe forms of finite-dimensional Lie algebras. They generalize Jordan algebras with identity involution and alternative algebras with involution. A structurable n-torus can be considered as a structurable analogue of the algebra of Laurent polynomials in n variables with graded involution. Allison and I first classified structurable 2-tori in [AY]. Then one of the experts in structurable theory, Faulkner, joined the project, and we have finished the classification of structurable tori in general [AFY]. Also, I previously classified quantum tori with involution in [Y3]. Quantum tori with involution can be considered as structurable tori which are associative algebras. My results about Jordan tori and quantum tori with involution play an important role in the classification of structurable tori. Quantum tori with involution also appear as coordinate algebras of EALAs of type C_r. Recently, Faulkner has classified the BC_2 type using the classification of the BC_1 type and a new notion, called a *quasi-structurable torus* in [F]. Thus, with the results about affine extensions by Neher [N1] (see also [N2]), the classification of EALAs is complete. However, the isomorphism theorem or conjugacy theorem of Cartan subalgebras of EALAs is still unknown.

Another interest of mine is to generalize EALAs in two directions. First I have considered the core of an EALA of type Δ as a special case of a *division* (Δ, G)*-graded Lie algebra* over a field of characteristic 0, where G is an arbitrary abelian group [Y2]. They have a Δ-grading with a compatible G-grading and satisfy a certain property with respect to the double grading. For example, a division (A_r, G)-graded Lie algebra for $r \geq 3$ is isomorphic to a central extension of $sl_{r+1}(A)$, where A is a division G-graded associative algebra. If G is the trivial group and A is central, then a division (Δ, G)-graded Lie algebra is simple and we obtain a generalization of the well-known finite-dimensional isotropic simple Lie algebras over a nonalgebraically closed field [Se]. For a division (Δ, G)-graded Lie algebra L and each root space L_μ $(\mu \in \Delta)$ of L which is G-graded, say $L_\mu = \oplus_{g\in G} L_\mu^g$, we define a subset S_μ of G as the support of the space, i.e., $S_\mu = \{g \in G \mid L_\mu^g \neq (0)\}$. We call the family of subsets S_μ of G indexed by Δ a *root system extended by* G. An interesting point is that these root systems are a natural generalization of extended affine root systems introduced in [Sa] and [AABGP]. An extended affine root system is considered as a root system extended by the group $\mathbb{Z}^n$ ([Az] and [Y5]). I have classified division $(\Delta, \mathbb{Z}^n)$-graded Lie algebras of type $\Delta = \mathrm{A}_r$ for $r \geq 2$, B_r for $r \geq 3$, D_r and E_r in [Y2], [Y4] and [Y5]. Also, a division (Δ, G)-graded Lie algebra is called a *Lie* G*-torus of type* Δ if the dimension of each homogenous space relative to the double grading is 0 or 1. We also say it a *Lie* n*-torus* or simply a *Lie torus* if $G = \mathbb{Z}^n$. I have shown that there exists a certain invariant bilinear form on a Lie torus in [Y6]. Using this result, one finds that a Lie torus coincides with the core of an EALA. Since the core itself of an EALA is important and Lie tori are defined by 3 rather simple axioms, this characterization will be useful in the further study of EALAs. Benkart an I studied division (C_r, G)-graded Lie algebras, and classified Lie G-torus of type C_r in [BY] using Seligman's techniques.

Centreless Lie n-tori are $\mathbb{Z}^n$-graded Lie algebras which are graded simple. Such algebras for $n = 1$, i.e., $\mathbb{Z}$-graded Lie algebras which are graded simple of finite growth, were classified by Mathieu in 1992 [M]. The classification of $\mathbb{Z}^n$-graded Lie algebras which are graded simple is a big open problem (see [OZ] for some results on $\mathbb{Z}^2$-graded Lie algebras with certain restrictions). For the classification of such graded Lie algebras, centreless Lie n-tori would serve as counterpart of loop (possibly twisted) algebras in the case $n = 1$. I would like to continue to study Lie tori from this point of view, too.

Another generalization of EALAs is the class of *locally extended affine Lie algebras*, which was recently introduced by Morita and myself in [MY]. These Lie algebras are closely related to locally finite split simple Lie algebras, studied by Neeb and Stume in [NS], and to locally finite root systems, studied by Loos and Neher in [LN1] (see also [LN2]). We classified locally extended affine Lie algebras of nullity 0. The classification of these Lie algebras for higher nullity seems very interesting.

Finally, the representation theory of EALAs is not much developed yet, but there are some papers in connection with the theory of vertex operators, e.g. [MRY], [T], [BB], [G] or [BL]. I would also like to study the representation theory of EALAs.

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