Evaluations of Multiple L-Values

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Abstract: In this paper, we generalize the methods of D. Zagier, J. Borwein, and R. Girgensohn for proving evaluations of multiple zeta values to the case of the multiple L-values. We also demonstrate numerical computation of these numbers based on the method of Crandall.

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Notation

In this paper, we will use the following notation:

$\mathbf{C}$	the set of complex numbers
$\mathbf{R}$	the set of real numbers
$\mathbf{Q}$	the set of rational numbers
$\mathbf{Z}$	the set of rational integers
$\mathbf{Z}_+$	the set of positive rational integers
μ_m	the set of complex m-th roots of unity
ζ_m	the primitive m-th root of unity $\exp(2\pi i/m)$
$\mathrm{cond}(\chi)$	the conductor of a Dirichlet character χ
$\mathrm{lcm}(a_1, \ldots, a_n)$	the least common multiple of the positive integers $a_1, \ldots, a_n$
$\gcd(a, b)$	the positive greatest common divisor of the integers a, b
ϕ	the Euler phi function

1. Introduction

The multiple zeta values of depth d of Euler-Riemann-Zagier are defined

$$\zeta(a_1, a_2, \ldots, a_d) = \sum_{0<n_1<n_2<\cdots<n_d} n_1^{-a_1} n_2^{-a_2} \cdots n_d^{-a_d}$$

where each $a_j \in \mathbf{Z}_+$, and $a_d > 1$. As mentioned in [2] and [5], these values have been related to such varied subjects as knot theory, cohomology of motives, and even quantum physics. Considering each a_j as a complex variable, J. Zhao[6] showed the existence of the meromorphic continuation of the multiple zeta function, which he obtained by multiplying by gamma factors. He also proved a region of absolute convergence, given by

$$\Re(a_d) > 1, \qquad \Re(a_1 + \cdots + a_d) > d$$

Thus, the sum defining a multiple zeta value for $a_d > 1$ converges absolutely. In this paper, each a_j will assume only positive integer values.

For Dirichlet characters $\chi_1, \ldots, \chi_d$, we define the multiple L-values of depth d

$$L\binom{\chi_1, \ldots, \chi_d}{a_1, \ldots, a_d} = \lim_{N\to\infty} \sum_{0<n_1<\cdots<n_d\le N} \frac{\chi(n_1)\cdots\chi_d(n_d)}{n_1^{a_1}\cdots n_d^{a_d}} \tag{1.1}$$

where each $a_j \in \mathbf{Z}_+$, if this limit converges. Zhao's domain of absolute convergence applies trivially to these values, also; thus, we have a region of absolute convergence for the multiple L-values.

We define the multiple polylogarithm of depth d

$$Li_{a_1,\ldots,a_d}(x_1, \ldots, x_d) = \lim_{N\to\infty} \sum_{0<n_1<\cdots<n_d\le N} \frac{x_1^{n_1}\cdots x_d^{n_d}}{n_1^{a_1}\cdots n_d^{a_d}} \tag{1.2}$$

for $a_j \in \mathbf{Z}_+$ and $x_j \in \mathbf{C}$, when this limit exists. For $d = 1$, this is the conventional polylogarithm $Li_a(x) = \sum_{n=1}^{\infty} x^n/n^a$. Note that if each x_j is a root of unity, Zhao's domain of absolute convergence applies to these values also. We will frequently use the finite Fourier expansion of a Dirichlet character χ of conductor D

$$\chi(n) = \sum_{k=1}^{D} c_k(\chi)\zeta_D^{nk} \tag{1.3}$$

where the $c_k(\chi)$ are complex numbers. We will use the fact that the $c_k(\chi)$, in fact, lie in the cyclotomic field obtained by adjoining ζ_D and the values of χ to $\mathbf{Q}$. This is because

$$c_k(\chi) = \frac{1}{D}\sum_{l=1}^{D} \chi(l)\zeta_D^{-kl} \tag{1.4}$$

which can be verified by substituting (1.4) into (1.3). For example, it can be shown that if χ is primitive, $c_k(\chi) = \bar{\chi}(k)/G(\bar{\chi})$, where $G(\psi)$ denotes the Gauss sum of the character ψ. In this paper, we will use (1.3) and (1.4) to write multiple L-values in terms of values of multiple polylogarithms at roots of unity.

We will use the notation $\chi \neq 1$ to indicate that χ is non-principal. Concerning the conditional convergence of some multiple L-values, we find the following:

Proposition 1. *The sum defining either of the following converges:*
i) A multiple polylogarithm (1.2) at roots of unity with $x_d \neq 1$
ii) A multiple L-value (1.1) with $\chi_d \neq 1$

Proof: We first show that the defining sum for a number as in i converges. If $a_d > 1$, we are in Zhao's domain of absolute convergence; thus, we can assume $a_d = 1$. We will show the sequence of partial sums in the last index n_d is Cauchy. We set

$$z(n_d) := \sum_{0<n_1<\cdots<n_{d-1}<n_d} \frac{x_1^{n_1}\cdots x_{d-1}^{n_{d-1}}}{n_1^{a_1}\cdots n_{d-1}^{a_{d-1}}}$$

and

$$c(n) := \sum_{m=1}^{n} x_d^m$$

Our assumptions gruarantee that $c(n)$ is bounded. Clearly,

$$x_d^{n_d} = c(n_d) - c(n_{d-1})$$

Consider for $N \in \mathbf{Z}_+$,

$$S_N = \sum_{n_d=1}^{N} z(n_d)\frac{x_d^{n_d}}{n_d} \tag{1.5}$$

We will show the sequence S_N is Cauchy in N. For $M, N \in \mathbf{Z}_+$, $M < N$,

$$\big|S_N - S_M\big| = \Big|\sum_{n_d=M+1}^{N} z(n_d)\frac{x_d^{n_d}}{n_d}\Big|$$

By Abel summation, this equals

$$\Big|\frac{z(N+1)c(N)}{N+1} + \sum_{n_d=M+1}^{N} c(n_d)\Big(\frac{z(n_d)}{n_d} - \frac{z(n_d+1)}{n_d+1}\Big)\Big| \tag{1.6}$$

The term in parentheses can be rewritten as

$$\sum_{0<n_1<\cdots<n_{d-1}<n_d} \frac{x_1 \cdots x_{d-1}^{n_{d-1}}}{n_1^{a_1} \cdots n_{d-1}^{a_{d-1}}}\Big(\frac{1}{n_d} - \frac{1}{n_d+1}\Big)$$

$$+ \sum_{0<n_1<\cdots<n_{d-1}=n_d} \frac{x_1^{n_1} \cdots x_{d-1}^{n_{d-1}}}{n_1^{a_1} \cdots n_{d-1}^{a_{d-1}}(n_d+1)}$$

Hence, the sum in (1.6) can be majorized by a sum $y_1 + y_2$, where each y_j is a difference of two partial sums of a multiple zeta value with last exponent > 1. Since the sequence of such partial sums is Cauchy, the sum in (1.6) approaches zero as $M, N \to \infty$.

To complete the proof that a sum as in i converges, it suffices to show $z(N)/N \to 0$ as $N \to \infty$. By inducting on d, and comparing the sum with an appropriate integral, one finds that $z(N)$ grows at most as a power of $\log(N)$.

To show that the defining sum for a number as in ii also converges, we define the partial sums

$$T_N = \sum_{0<n_1<\cdots<n_d\le N} \frac{\chi_1(n_1)\cdots\chi_d(n_d)}{n_1^{a_1} \cdots n_{d-1}^{a_{d-1}} n_d}$$

We now use (1.3) and (1.4) to write this as a **C**-linear combination of the quantities (1.5), each of which we have already shown to converge as $N \to \infty$. Terms with $x_d = 1$ cannot appear in this combination, since, in (1.3), for $\chi \neq 1$, $c_D(\chi) = 0$, where $D = \operatorname{cond}(\chi)$. Thus, the sequence $\{T_N\}$ also converges as $N \to \infty$. This finishes the proof of the proposition. //

Therefore, we know the multiple L-values include the limits (1.1) for which $a_d > 1$, and also those for which $\chi_d \neq 1$ and $a_d = 1$.

Many evaluations of multiple zeta values have been found (some of which were known to Euler). Some examples are:

$$\zeta(1,2) = \zeta(3) \qquad \zeta(1,3) = \frac{3}{2}\zeta(4) - \frac{1}{2}\zeta(2)^2$$

$$\zeta(1,4) = 2\zeta(5) - \zeta(2)\zeta(3) \qquad \zeta(1,1,2) = \zeta(4)$$

$$\zeta(2,3) = \zeta(1,2,2) = -\frac{11}{2}\zeta(5) + 3\zeta(2)\zeta(3)$$

$$\zeta(2,1,5) = \frac{157}{360}\zeta(8) + \frac{5}{2}\zeta(3)\zeta(5) - \frac{3}{2}\zeta(3)^2\zeta(2) + \frac{2}{5}\zeta(3,5) \tag{1.7}$$

We will give examples of similar evaluations for double L-values.

In the study of multiple zeta values, the terms *evaluate* and *reduce* have been used as follows. A multiple zeta value of depth d is said to *evaluate* if it can be written in terms of depth one zeta values; it is said to *reduce* if it can be written in terms of multiple zeta values of depth strictly less than d. We will retain the use of these terms in the setting of the multiple L-values.

We make the following definition:

Definition. *Let $\chi_1, \ldots, \chi_d$ be Dirichlet characters, and $x_1, \ldots, x_d$ be roots of unity. We define the weight of*

$$\zeta(a_1, \ldots, a_d) \quad \text{or} \quad L\binom{\chi_1, \ldots, \chi_d}{a_1, \ldots, a_d} \quad \text{or} \quad Li_{a_1,\ldots,a_d}(x_1, \ldots, x_d)$$

to be $a_1 + \cdots + a_d$. Further, suppose $y_1, \ldots, y_k$ each have weight w, and z has weight v. Then we define the weight of $y_1 + \cdots + y_k$ to also be w and the weight of $y_1 z$ to be $v + w$.

The consistency of this definition is still conjectural. Notice that all of our listed evaluations respect this notion of weight. In fact, so also do all known evaluations of multiple zeta values.

We also make the following definition.

Definition. *For positive integers D, m, and j, we denote by $R^j_{D,m}$ the ring generated by $\mathbf{Q}(\zeta_m)$ and the values of multiple polylogarithms of depth $\leq j$ at D-th roots of unity.*

D. Zagier[5] showed:

Theorem. *Let $a, b \in \mathbf{Z}_+$, with $b > 1$. If $a + b$ is odd, then $\zeta(a,b)$ lies in $R^1_{1,1}$.*

J. Borwein and R. Girgensohn[1] proved the analogue of this result for the triple zeta function, which can be stated as follows.

Theorem. *Let $a, b, c \in \mathbf{Z}_+$, with $c > 1$. If $a + b + c$ is even, then $\zeta(a,b,c)$ lies in $R^2_{1,1}$.*

We will generalize these theorems to the double L-values and three special cases of the triple L-values. We will adapt the methods of [1] to prove the following:

Theorem 1. *Let χ (resp., ψ) be Dirichlet characters of conductor D (resp., E), and $a, b \in \mathbf{Z}_+$, with $a + b \geq 3$. Also suppose that either $b > 1$ or $\psi \neq 1$. Let $m = \operatorname{lcm}(D, E, \phi(D), \phi(E))$ and $F = \operatorname{lcm}(D, E)$. If $\chi(-1)\psi(-1) = (-1)^{a+b-1}$, then*

$$L\binom{\chi, \psi}{a, b} \in R^1_{F,m}$$

Corollary. *With hypotheses as in theorem 1, $L(\chi, \psi; a, b)$ can be written as a finite sum $\sum a_j l_j m_j$, where each l_j and m_j is either an L-series value or a value of Li_1 at an F-th root of unity, and each $a_j \in \mathbf{Q}(\zeta_m)$.*

Theorem 2. *Let χ be one of the quadratic characters of conductor 3, 4, or 5, D the conductor of χ, and $m = \operatorname{lcm}(D, \phi(D))$. Let $a, b, c \in \mathbf{Z}_+$, with $a + b + c \geq 4$. If χ has the same parity as $a + b + c$, then any of*

$$L\binom{\chi, 1, 1}{a, b, c}, \qquad L\binom{1, \chi, 1}{a, b, c}, \qquad L\binom{1, 1, \chi}{a, b, c}$$

(where in the first two cases $c > 1$) lies in $R^2_{D,m}$.

2. Evaluations of Double L-Values

Let χ_D be the quadratic character of conductor D. Some evaluations of double L-values which can be found using the methods of this section are:

$$L\binom{\chi_3, 1}{1, 3} = \frac{1}{2}\log 3 L_{\chi_3}(3) + \frac{13}{9}L_{\chi_3}(1)\zeta(3) - \frac{1}{3}L_{\chi_3}(2)\zeta(2) - L_{\chi_3}(4) \tag{2.1}$$

$$L\binom{\chi_3, 1}{2, 2} = L_{\chi_3}(4) + \frac{4}{3}\zeta(2)L_{\chi_3}(2) - \frac{26}{9}L_{\chi_3}(1)\zeta(3) \tag{2.2}$$

$$\begin{aligned} L\binom{\chi_3, 1}{1, 5} &= \frac{1}{2}\log 3 L_{\chi_3}(5) + \frac{121}{81}L_{\chi_3}(1)\zeta(5) - \frac{13}{27}L_{\chi_3}(2)\zeta(4) \\ &+ \frac{4}{9}L_{\chi_3}(3)\zeta(3) - \frac{27}{81}L_{\chi_3}(4)\zeta(2) - L_{\chi_3}(6) \end{aligned} \tag{2.3}$$

$$\begin{aligned} L\binom{\chi_3, 1}{2, 4} &= 2L_{\chi_3}(6) - \frac{484}{81}L_{\chi_3}(1)\zeta(5) + \frac{40}{27}L_{\chi_3}(2)\zeta(4) \\ &- \frac{8}{9}L_{\chi_3}(3)\zeta(3) + L_{\chi_3}(4)\zeta(2) \end{aligned} \tag{2.4}$$

$$\begin{aligned} L\binom{\chi_3, 1}{3, 3} &= -\frac{11}{2}L_{\chi_3}(6) - \zeta(2)L_{\chi_3}(4) + \frac{13}{9}\zeta(3)L_{\chi_3}(3) \\ &+ \frac{242}{27}L_{\chi_3}(1)\zeta(5) \end{aligned} \tag{2.5}$$

$$\begin{aligned} L\binom{\chi_3, 1}{4, 2} &= \frac{9}{2}L_{\chi_3}(6) - \frac{484}{81}L_{\chi_3}(1)\zeta(5) - 2L_{\chi_3}(3)\zeta(3) \\ &+ \frac{4}{3}L_{\chi_3}(4)\zeta(2) \end{aligned} \tag{2.6}$$

$$\begin{aligned} L\binom{\chi_5, 1}{1, 2} &= L_{\chi_5}(1)\zeta(2) - \frac{1}{12}(r_1 + 4r_2 + 7r_3)L_{\chi_5}(2) \\ -\frac{1}{4}\frac{\bar{G}_o}{G_e}(r_1 - ir_2 + ir_3 - r_4)L_{\chi_o}(2) &+ \frac{1}{4}\frac{G_o}{G_e}(r_1 + ir_2 - ir_3 - r_4)\bar{L}_{\chi_o}(2) \end{aligned} \tag{2.7}$$

In the last expression, $r_j = \log(1 - \zeta^j)$, with $\zeta = \exp(2\pi i/5)$, G_e is the Gauss sum of χ_5, χ_o is an odd character of conductor 5 such that $\chi_o(2) = i$, and G_o its Gauss sum. The last expression is also an example of an evaluation for which not all of the coefficients are in $\mathbf{Q}$. Note each of these evaluations is weight-preserving. Most of these identities were discovered numerically, using the series in section 5, before a proof was forthcoming. At the end of this section, we will derive the identities (2.1) and (2.2).

A partial fractions decomposition is

$$\frac{1}{m^a(m-n)^b} = \sum_{c+d=a+b}\left\{M_{c,d}^{a,b}\frac{1}{m^c n^d} + N_{c,d}^{a,b}\frac{1}{n^c(m-n)^d}\right\} \tag{2.8}$$

where c and d are restricted to the positive integers, and

$$M_{c,d}^{a,b} = (-1)^b\binom{d-1}{b-1} \qquad N_{c,d}^{a,b} = (-1)^{a+c}\binom{c-1}{a-1} \tag{2.9}$$

This is a result due to Markett[4], but in fact was known to Euler (see [1]). We will substitute (2.8) into expressions involving double sums in order to find linear combinations of these double sums which equal some linear combination of products of single sums.

We define the following extension of binomial coefficients:

Definition. *For arbitrary integers a, b, we define the binomial coefficient*

$$\binom{a}{b} = \begin{cases} \frac{a(a-1)\cdots(b+1)}{(a-b)!}, & a \ge b \\ 0, & \textit{otherwise} \end{cases}$$

We will use the following to derive many matrix identities throughout this paper.

Proposition 2. *For non-negative integers a, b, c, d,*

$$\sum_{k=-\infty}^{\infty}(-1)^k\binom{a-k}{b}\binom{c}{d+k}=(-1)^d\binom{a+d-c}{b-c} \tag{2.10}$$

Proof: Note the above is a finite sum. We have for $e\in\mathbf{Z}$ and $f\in\mathbf{Z}_+$ the power series expansions

$$\sum_{k=e}^{\infty}\binom{k}{e}x^k=\frac{x^e}{(1-x)^{e+1}}\qquad\sum_{k=0}^{f}(-1)^k\binom{f}{k}x^k=(1-x)^f \tag{2.11}$$

The second of these is standard, and the first arises from differentiation of geometric series for $e\geq 0$. To show that the first holds for all $e\in\mathbf{Z}$, note that for $e<0$,

$$\sum_{k=e}^{\infty}\binom{k}{e}x^k=\sum_{k=e}^{0}\binom{k}{e}x^k=\sum_{k=0}^{-e}\binom{-k}{e}x^{-k} \tag{2.12}$$

for $e<0$. By definition,

$$\binom{-k}{e}=\frac{-k(-k-1)\cdots(e+1)}{(-k-e)!}$$

for $0\leq k\leq -e$. This equals

$$(-1)^{k+e}\frac{k(k+1)\cdots(-e-1)}{(-k-e)!}=(-1)^{k+e}\binom{-e-1}{k-1} \tag{2.13}$$

Substituting (2.13) into (2.12) gives

$$\sum_{k=0}^{-e}(-1)^{k+e}\binom{-e-1}{k-1}x^{-k}=\frac{(-1)^{e-1}}{x}\sum_{k=0}^{-e}\binom{-e-1}{k-1}(-x)^{-k+1}$$

$$=\frac{(-1)^{e-1}}{x}(1-\frac{1}{x})^{-e-1}=\frac{(-1)^{e-1}}{x}\frac{(x-1)^{-e-1}}{x^{-e-1}}=\frac{x^e}{(1-x)^{e+1}}$$

Thus, the first power series expansion in (2.11) does indeed hold for all integers e. Now, the desired sum (2.10) is $(-1)^d$ times the coefficient of x^{a+d} in the power series expansion of

$$\frac{x^b}{(1-x)^{b-c+1}}=x^c\frac{x^{b-c}}{(1-x)^{b-c+1}}$$

which is as stated. //

The main result of this section is the following:

Theorem 1. *Let χ (resp. ψ) be a Dirichlet character of conductor D (resp. E), and $a,b\in\mathbf{Z}_+$ with $a+b\geq 3$. In addition, suppose $b>1$ or $\psi\neq 1$. Set $m=\operatorname{lcm}(D,E,\phi(D),\phi(E))$ and $F=\operatorname{lcm}(D,E)$. If $\chi(-1)\psi(-1)=(-1)^{a+b-1}$, then*

$$L\binom{\chi,\psi}{a,b}\in R^1_{F,m}$$

.

Proof: We define for Dirichlet characters χ and ψ and positive integers a,b,N the partial sums

$$L_N\binom{\chi,\psi}{a,b}=\sum_{0<m<n\leq N}\frac{\chi(m)\psi(n)}{m^an^b}\quad K_N\binom{\chi,\psi}{a,b}=\sum_{0<m<n\leq N}\frac{\chi(m)\psi(n-m)}{m^an^b}$$

$$L_N(\chi, a) = \sum_{n=1}^{N} \frac{\chi(n)}{n^a} \qquad Li_a^N(x) = \sum_{n=1}^{N} \frac{x^n}{n^a} \tag{2.14}$$

In deriving the identities listed at the first of this section, we need to use the fact that

$$\lim_{N\to\infty} Li_1^N(x) = \log(1-x) \tag{2.15}$$

for $|x| = 1$, $x \neq 1$, where $\log z$ denotes the principal branch of the logarithm

$$\frac{-i\pi}{2} < \text{Arg}\big(\log z\big) < \frac{i\pi}{2}$$

To show (2.15), note that by definition, $\lim_{N\to\infty} Li_1^N(x) = \sum_{n=1}^{\infty} x^n/n$. Since this sum converges, it equals

$$\lim_{\substack{y\to x\\ |y|<1}} \log(1-y) = \log(1-x)$$

by Abel's theorem, because the principle branch of the logarithm is continuous at $1-x$. The corollary to theorem 1 requires the following:

Lemma 1. *For* $\chi \neq 1$,

$$\lim_{N\to\infty} L_N(\chi, 1) = L(\chi, 1) \tag{2.16}$$

Proof: For $\Re(s) > 1$ we can write

$$L(\chi, 1) = \sum_{n=1}^{N} \frac{\chi(n)}{n^s} + \sum_{n=N+1}^{\infty} \frac{\chi(n)}{n^s}$$

since such s are in the domain of absolute convergence. Set $S(x) = \sum_{0<n\leq x} \chi(n)$. By elementary theory of Dirichlet characters, $S(x)$ is periodic. Since $S(x)$ is constant on intervals $[n, n+1)$ for $n \in \mathbf{Z}$, $S(x)$ assumes only finitely many values on any period. Therefore, $S(x)$ is bounded. Applying Abel summation, we see

$$\sum_{n=N+1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=N+1}^{\infty} \big(S(n) - S(N)\big)\big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\big)$$

$$= s \sum_{n=N+1}^{\infty} \int_{n}^{n+1} \frac{S(x) - S(N)}{x^{s+1}}\, dx = s \int_{N+1}^{\infty} \frac{S(x) - S(N)}{x^{s+1}}\, dx \tag{2.17}$$

This integral converges uniformly on compact subsets of the right half plane $\Re(s) > 0$, and hence defines an analytic function there. So (2.17) gives a formula for $L(\chi, s)$ in this region also, for any choice of $N \in \mathbf{Z}_+$. Since $S(x)$ is bounded, the absolute value of the integral is bounded by a positive constant times $(N+1)^{-\Re(s)}$. Hence, for $\Re(s) > 0$, the value of the integral approaches zero as $N \to \infty$. This proves the lemma. //

Let us also define for Dirichlet characters χ and ψ and for $a, b \in \mathbf{Z}_+$ the values

$$K\binom{\chi, \psi}{a, b} = \lim_{N\to\infty} K_N\binom{\chi, \psi}{a, b} = \sum_{0<m<n} \frac{\chi(m)\psi(n-m)}{m^a n^b} \tag{2.18}$$

when these limits exist (which they do if either $b > 1$ or $\psi \neq 1$, as is shown by applying (1.3), (1.4), and proposition 1).

The strategy of the proof is as follows. We will find $\mathbf{Z}$-linear combinations of the double sums L_N and K_N which equal $\mathbf{Q}(\zeta_m)$-linear combinations of products of single sums, plus possibly some error terms. This will be done using three evaluation techniques which are each weight-preserving. In lemma 3, it will be shown that the error terms approach zero as $N \to \infty$. The coefficients of the $\mathbf{Z}$-linear combinations will form a matrix, which will be shown to have full rank when the weight has the appropriate parity. Thus, we evaluate each L_N and K_N double sum as a linear combination of products of single sums, plus an expression which approaches zero as $N \to \infty$. At the end of the proof, we will resolve the issues concerning what happens when we let $N \to \infty$ in these evaluations.

The first of our three evaluation techniques consists of applications of the "shuffle" relation. This says that for $a, b \in \mathbf{Z}_+$,

$$L_N\binom{\chi, \psi}{a, b} + L_N\binom{\psi, \chi}{b, a} + \sum_{n=1}^{N} \frac{\chi(n)\psi(n)}{n^{a+b}} = L_N(\chi, a)L_N(\psi, b) \tag{2.19}$$

We use the finite Fourier expansion (1.3) and (1.4) to write the right-hand side and the last term on the left-hand side as a $\mathbf{Q}(\zeta_m)$-linear combination of products of Li_f^N values at F-th roots of unity for $f \in \mathbf{Z}_+$. Thus, we have $\mathbf{Z}$-linear combinations of L_N double sums which equal linear combinations of products of single sums. Note that there are no error terms.

Remark: Letting $N \to \infty$ in (2.19), we also see

$$L\binom{\chi, \psi}{a, b} + L\binom{\psi, \chi}{b, a} + \sum_{n=1}^{\infty} \frac{\chi(n)\psi(n)}{n^{a+b}} = L(\chi, a)L(\psi, b) \tag{2.20}$$

when the sum defining each term converges.

In order to obtain our second evaluation technique, we write

$$L_N\binom{\chi, \psi}{a, b} = \sum_{n=1}^{N}\sum_{m=1}^{n-1} \frac{\chi(m)\psi(n)}{m^a n^b} = \sum_{n=1}^{N}\sum_{m=1}^{n-1} \frac{\chi(n-m)\psi(n)}{(n-m)^a n^b} \tag{2.21}$$

Using (2.8) and (2.9), this equals

$$\begin{aligned} \sum_{\substack{c+d=a+b \\ c,d>0}} \Big\{ M_{c,d}^{b,a} \sum_{n=1}^{N}\sum_{m=1}^{n-1} \frac{\chi(n-m)\psi(n)}{m^d n^c} \\ + N_{c,d}^{b,a} \sum_{n=1}^{N}\sum_{m=1}^{n-1} \frac{\chi(n-m)\psi(n)}{m^c (n-m)^d} \Big\} \end{aligned} \tag{2.22}$$

The first double sum in the braces here equals

$$\sum_{0<m<n\le N} \frac{\chi(n-m)\psi(n)}{m^d n^c} = \sum_{0<n<m\le N} \frac{\psi(m)\chi(m-n)}{m^c n^d}$$

$$= \chi(-1)\Big\{ \sum_{m,n=1}^{N} \frac{\psi(m)\chi(n-m)}{m^c n^d} - \delta_{\chi,1} L_N(\psi, c+d)$$

$$- \sum_{0<m<n\le N} \frac{\psi(m)\chi(n-m)}{m^c n^d} \Big\}$$

$$= \chi(-1)\Big\{ \sum_{m,n=1}^{N} \frac{\psi(m)\chi(n-m)}{m^c n^d} - \delta_{\chi,1} L_N(\psi, c+d) - K_N\binom{\psi, \chi}{c, d} \Big\} \tag{2.23}$$

where we define for two Dirichlet characters χ and ψ

$$\delta_{\chi,\psi} = \begin{cases} 1, & \chi = \psi \\ 0, & \text{otherwise} \end{cases}$$

The second double sum in braces in (2.22) equals

$$\sum_{n=1}^{N}\sum_{m=1}^{N-n}\frac{\chi(m)\psi(m+n)}{m^d n^c} = \sum_{m,n=1}^{N}\frac{\chi(m)\psi(m+n)}{m^d n^c} - e_1^N\binom{\chi,\psi}{d,c} \tag{2.24}$$

where we have the error term

$$e_1^N\binom{\chi,\psi}{x,y} = \sum_{n=1}^{N}\sum_{m=N-n+1}^{N}\frac{\chi(m)\psi(m+n)}{m^x n^y} \tag{2.25}$$

In lemma 3, we will show all error terms which can appear approach zero as $N \to \infty$; hence, they may be diregarded. We first use (1.3) and (1.4) to express the first double sum on the right-hand side in each of (2.23) and (2.24) as a $\mathbf{Q}(\zeta_m)$-linear combination of products of single Li_f^N sums, evaluated at F-th roots of unity, for $f \in \mathbf{Z}_+$. Thus, we express each $L_N(\chi, \psi; a, b)$ as a $\mathbf{Z}$-linear combination of the numbers $K_N(\psi, \chi; c, d)$ (where c, d range over the positive integers with $c + d = a + b$) plus a $\mathbf{Z}$-linear combination of products of single-sum L_N values, plus a $\mathbf{Q}(\zeta_m)$-linear combination of single sums, plus an expression which approaches zero as $N \to \infty$. Hence, we have linear combinations of double L_N- and K_N-values which equal linear combinations of products of single sums, plus an expression which approaches zero as $N \to \infty$.

In order to obtain our last evaluation technique, we apply the same procedure to the K_N sums:

$$\begin{aligned} K_N\binom{\chi,\psi}{a,b} &= \sum_{n=1}^{N}\sum_{m=1}^{n-1}\frac{\chi(m)\psi(n-m)}{m^a n^b} = \sum_{n=1}^{N}\sum_{m=1}^{n-1}\frac{\chi(n-m)\psi(m)}{(n-m)^a n^b} \\ &= \sum_{\substack{c+d=a+b\\ c,d>0}}\Big\{M_{c,d}^{b,a}\sum_{n=1}^{N}\sum_{m=1}^{n-1}\frac{\chi(n-m)\psi(m)}{m^d n^c} \\ &\qquad + N_{c,d}^{b,a}\sum_{m=1}^{N}\sum_{m=1}^{n-1}\frac{\chi(n-m)\psi(m)}{m^c (n-m)^d}\Big\} \end{aligned} \tag{2.26}$$

The first double sum in braces equals

$$\sum_{0<m<n\le N}\frac{\chi(n-m)\psi(m)}{m^d n^c} = K_N\binom{\psi,\chi}{d,c}$$

while the second is

$$\begin{aligned} \sum_{n=1}^{N}\sum_{m=1}^{N-n}\frac{\psi(m)\chi(n)}{m^c n^d} &= \sum_{m,n=1}^{N}\frac{\psi(m)\chi(n)}{m^c n^d} - e_2^N\binom{\psi,\chi}{c,d} \\ &= L_N(\psi,c)L_N(\chi,d) - e_2^N\binom{\psi,\chi}{c,d} \end{aligned} \tag{2.27}$$

where

$$e_2^N\binom{\chi,\psi}{x,y} = \sum_{n=1}^{N}\sum_{m=N+1-n}^{N}\frac{\chi(m)\psi(n)}{m^x n^y} \tag{2.28}$$

Thus, we express each value $K_N(\chi, \psi; a, b)$ as a $\mathbf{Z}$-linear combination of values $K_N(\psi, \chi; c, d)$ (where c, d range over the positive integers with $c + d = a + b$) plus a $\mathbf{Z}$-linear combination of products of single-sum L_N-values, plus some error terms. We again use (1.3) and (1.4) to express the single-sum L_N-values on the right side of (2.27) in terms of Li_f^N values at F-th roots of unity, for $f \in \mathbf{Z}_+$. This writes each K_N value as a $\mathbf{Z}$-linear combination of K_N values, plus a $\mathbf{Q}(\zeta_m)$-linear combination of products of Li_f^N values at F-th roots of unity, plus some error terms. Thus, we get linear combinations of K_N-values which equal linear combinations of single sums, plus error terms.

Remark: By letting $N \to \infty$ in (2.21) and (2.26), we find linear combinations of L_N and K_N double sums which equal linear combinations of products of (single) polylogarithms evaluated at roots of unity (analogous to (2.20)), when each involved sum converges.

Set $w = a + b$ and $v = w - 1$. Applying (2.19) and (2.22) through (2.28), we now have three methods of obtaining linear combinations of the $4v$ values

$$L_N\binom{\chi,\psi}{c,d}, \quad L_N\binom{\psi,\chi}{c,d}, \quad K_N\binom{\chi,\psi}{c,d}, \quad K_N\binom{\psi,\chi}{c,d} \tag{2.29}$$

(where $c + d = a + b$, $c, d \in \mathbf{Z}_+$) which equal linear combinations of single sums, plus error terms.

For $j = 1, \dots, v$, set $p_j = (j, w - j) \in \mathbf{Z}_+ \times \mathbf{Z}_+$. We now order the values in (2.29) as follows:

$$\begin{gathered} L_N\binom{\chi,\psi}{p_1}, \dots, L_N\binom{\chi,\psi}{p_v}, L_N\binom{\psi,\chi}{p_1}, \dots, L_N\binom{\psi,\chi}{p_v}, \\ \dots, K\binom{\chi,\psi}{p_1}, \dots, K_N\binom{\chi,\psi}{p_v}, \dots, K_N\binom{\psi,\chi}{p_1}, \dots, K_N\binom{\psi,\chi}{p_v} \end{gathered} \tag{2.30}$$

The partial fractions expansions (2.21) through (2.23) and (2.26) through (2.28) for the double sums L_N and K_N give $4v$ linear relations among these numbers, disregaring error terms (which, as mentioned before, in lemma 3 will be shown to approach zero, as $N \to \infty$). The shuffle relation (2.19) gives another v relations among the double-sum L_N's. We order these $5v$ relations by first listing the $2v$ relations resulting from applying (2.21) through (2.23) to the first $2v$ numbers in (2.30) in the order given in (2.30). We then list the $2v$ relations resulting from applying (2.26) through (2.28) to the last $2v$ numbers in (2.30) also in the order given in (2.30). Finally we list the shuffle relations (2.19) applied to the first v numbers in (2.30), also in the order given there.

We will next construct a matrix of the coefficients of these linear combinations (ignoring the error terms and the linear combinations of products of single sums which appear), and show that when $\chi(-1)\psi(-1) = (-1)^{w-1}$, this matrix has full rank. The square identity matrix of appropriate dimension will be denoted by I. We will denote by X_{ij} the ij-entry of a $D \times E$ matrix X, for $1 \le i \le D$ and $1 \le j \le E$. We define a $5v \times 4v$ matrix M by declaring M_{ij} to be the coefficient in the ith relation (using the ordering of the relations given above) of the jth number in the list (2.30). As a block matrix, by (2.19), (2.22), (2.23), (2.26), and (2.27), this matrix takes the form

$$M = \begin{pmatrix} I & & & \chi(-1)B \\ & I & \psi(-1)B & \\ & & I & A \\ & & A & I \\ I & P & & \end{pmatrix} \tag{2.31}$$

for some $v \times v$ matrices A and B, where P is the $v \times v$ permutation matrix with 1's along the anti-diagonal. We use (2.9), (2.22), (2.23) and (2.26) to compute the matrices A and B:

$$A_{ij} = -M^{b,a}_{c,d} = (-1)^{a-1}\binom{d-1}{a-1} = (-1)^{i-1}\binom{j-1}{i-1} \tag{2.32}$$

$$B_{ij} = M^{b,a}_{c,d} = (-1)^{a}\binom{d-1}{a-1} = (-1)^{i}\binom{w-j-1}{i-1} = (-AP)_{ij} \tag{2.33}$$

In (2.32), we set $j = d$, while in (2.33), we set $j = c$. This difference is due to the fact that using (2.22) to relate the values in (2.29) involved using a shuffle relation, while using (2.26) to do the same did not. Thus, (2.31) equals

$$\begin{pmatrix} I & & & -\chi(-1)AP \\ & I & -\psi(-1)AP & \\ & & I & A \\ & & A & I \\ I & P & & \end{pmatrix} \tag{2.34}$$

where A is defined by (2.32).

Note

$$(A^2)_{ij} = (-1)^{i-1} \sum_{k=1}^{w-1} (-1)^{k-1} \binom{k-1}{i-1} \binom{j-1}{k-1}$$

$$= (-1)^{i-1} \sum_{k=0}^{w-2} (-1)^{k} \binom{k}{i-1} \binom{j-1}{j-k-1}$$

By proposition 2, this is

$$(-1)^{i+j} \binom{0}{i-j} = I_{ij}$$

Thus,

$$A^2 = I \tag{2.35}$$

It is elementary to see that $P^2 = I$. A crucial fact is

Lemma 2.

$$APA = (-1)^w PAP \tag{2.36}$$

Proof: We write

$$(AP)_{ij} = \sum_{k=1}^{w-1} (-1)^{i-1} \binom{k-1}{i-1} \delta_{k+j,w} = (-1)^{i-1} \binom{w-j-1}{i-1}$$

where for integers x, y,

$$\delta_{x,y} = \begin{cases} 1, & x = y \\ 0, & \text{otherwise} \end{cases}$$

Then

$$(APA)_{ij} = (-1)^{i-1} \sum_{k=1}^{w-1} (-1)^{k-1} \binom{w-k-1}{i-1} \binom{j-1}{k-1}$$

$$= (-1)^{i-1} \sum_{k=0}^{w-2} (-1)^{k} \binom{w-k-2}{i-1} \binom{j-1}{k}$$

while

$$(PAP)_{ij} = \sum_{k=1}^{w-1} \delta_{i+k,w} (-1)^{k-1} \binom{w-j-1}{k-1}$$

$$= (-1)^{w-i-1} \binom{w-j-1}{w-i-1} = (-1)^{w-i-1} \binom{w-j-1}{i-j}$$

Applying proposition 2 now gives the result. //

Our goal is to show the block matrix (2.34) has full rank when w has parity as in the hypothesis of the theorem. Note that multiplying the fourth row on the left by A gives the third row; thus, the fourth row is redundant. Eliminating the fourth row, we define the remaining matrix

$$M_1 = \begin{pmatrix} I & & & -\chi(-1)AP \\ & I & -\psi(-1)AP & \\ & & I & A \\ I & P & & \end{pmatrix} \tag{2.37}$$

One sees, using the identities (2.35) and (2.36), that M_1^{-1} equals

$$\frac{1}{2} \begin{pmatrix} I & -P & -\psi(-1)PAP & I \\ -P & I & \psi(-1)AP & P \\ -\psi(-1)PAP & -\psi(-1)PA & I & \psi(-1)PAP \\ -\chi(-1)PA & -\chi(-1)PAP & A & \chi(-1)PA \end{pmatrix} \tag{2.38}$$

when $\chi(-1)\psi(-1) = (-1)^{w-1}$.

Remark: The above expression for M_1^{-1} can be used to find explicit formulas for the evaluations promised by the theorem. E.g., in the case of the double zeta values,

$$\zeta(k, w-k) = \frac{1+(-1)^{k-1}}{2}\zeta(k)\zeta(w-k) - \frac{1}{2}\Big\{1 + (-1)^k\Big(\binom{w-1}{k} + \binom{w-1}{k-1}\Big)\Big\}\cdot$$

$$\zeta(w) + (-1)^k \sum_{\substack{j=1 \\ j \text{ odd}}}^{w-1} \Big\{\binom{j-1}{k-1} + \binom{j-1}{w-k-1}\Big\}\zeta(j)\zeta(w-j) \tag{2.39}$$

when $0 < k < w$ and w is odd. More generally, when $\chi(-1)\psi(-1) = (-1)^{w-1}$,

$$L\binom{\chi, \psi}{k, w-k} = \frac{1}{2}\Big\{(-1)^k \sum_{l=1}^{w-1}\Big\{\chi(-1)\binom{l-1}{k-1}\big(S^-_{w-l}(\psi, \chi) - \delta_{\chi,1}L_\psi(w)\big)$$

$$+(-1)^{w+l}\binom{l-1}{w-k-1}S^+_{w-l}(\chi, \psi)\Big\} - (-1)^{w+k}\sum_{l=1}^{w-1}\Big\{\psi(-1)\cdot$$

$$\binom{l-1}{w-k-1}\big(S^-_{w-l}(\chi, \psi) - \delta_{\psi,1}L_\chi(w)\big) + (-1)^{w+l}\binom{l-1}{k-1}S^+_{w-l}(\psi, \chi)\Big\}$$

$$+\big(1 + \psi(-1)(-1)^{k+w}\big)L_\chi(k)L_\psi(w-k) - \sum_{n=1}^{\infty}\frac{\chi(n)\psi(n)}{n^w}\Big\} \tag{2.40}$$

where for two Dirichlet characters σ, τ and $w \in \mathbf{Z}$, $w \geq 3$, we define

$$S^-_j(\sigma, \tau) = \sum_{m,n=1}^{\infty}\frac{\sigma(m)\tau(n-m)}{m^j n^{w-j}}$$

$$S^+_j(\sigma, \tau) = \sum_{m,n=1}^{\infty}\frac{\sigma(m)\tau(m+n)}{m^j n^{w-j}}$$

We will now prove an important lemma referenced earlier in the proof.

Lemma 3. *Let e_1^N and e_2^N be as given in (2.25) and (2.28). Then*

$$\lim_{N\to\infty} e_1\binom{x, y}{c, d} = \lim_{N\to\infty} e_2\binom{x, y}{c, d} = 0$$

where $c, d \in \mathbf{Z}_+$, $c + d \geq 3$.

Proof: In each case, we majorize by taking absolute values inside the summation. It is enough to consider the cases $c = 1$, $d = 2$, and $c = 2$, $d = 1$. Making use of symmetry, we need only show that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\sum_{m=N+1-n}^{N}\frac{1}{m^2 n} = 0 \tag{2.41}$$

By estimating the inner sum, the sum in (2.41) is

$$O\Big\{\frac{1}{N}\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=1}^{N}\frac{1}{n(N+1-n)}\Big\} = O\Big\{\frac{\log N}{N} + \frac{1}{N+1}\sum_{n=1}^{N}\Big(\frac{1}{n} + \frac{1}{N+1-n}\Big)\Big\}$$

by applying partial fractions. Estimating the last sum, we obtain $O(\log N/N)$ for this sum also. Thus, the limit (2.41) is zero, as required. //

Hence, all error terms which appear approach zero as $N \to \infty$. Hence, we can assume the evaluations of the numbers (2.29) resulting from solving the system of equations from which the matrix M was constructed are $\mathbf{Q}(\zeta_m)$-linear combinations of products of partial polylogarithms $Li_f^N(\zeta)$ for $f \in \mathbf{Z}_+$ and $\zeta \in \mu_F$.

We now show that these evaluations actually are correct. It can happen that sums appear in the evaluations which do not converge as $N \to \infty$. We must show that the summands in which these non-convergent sums appear approach a number which equals an element of $R^1_{F,m}$. It will actually be shown that such summands approach zero as $N \to \infty$. We define

$$\zeta_N(1) = \sum_{n=1}^{N} \frac{1}{n}$$

for x a root of unity. Let S_N be one of the values in (2.29), which converges as $N \to \infty$ by either Zhao's domain of absolute convergence, or by applying (1.3), (1.4), and proposition 1. We write the evaluation of S_N resulting from solving the system of equations from which M was constructed (ignoring the error terms) in the form

$$S_N = \rho_0(N) + \rho_1(N)\zeta_N(1) \tag{2.42}$$

where for $j = 1, 2$, $\rho_j(N)$ is a $\mathbf{Q}(\zeta_m)$-linear combination of partial polylogarithms $Li_f^N(x)$ for F-th roots of unity x and $f \in \mathbf{Z}_+$, and each term has either $f = 1$ and $x \neq 1$, or $f > 1$ (so that each term converges as $N \to \infty$). Since we only consider weights $w \geq 3$ in theorem 1, we do not here consider the term $\zeta_N(1)^2$, which could only result from evaluating a number of weight 2, because each of our evaluation techniques is weight-preserving. Now, dividing both sides of (2.42) by $\zeta_N(1)$ shows $\rho_1(N) \to 0$ as $N \to \infty$ (since the other terms do). In fact, our proof shows that $\rho_1(N)$ is a finite linear combination $\sum_j \alpha_j Li_c^N(x_j)$ where each $\alpha_j \in \mathbf{Q}(\zeta_m)$, $x_j \in \mu_F$, and $c = w - 1$. Since $w \geq 3$, each summand of $\rho_1(N)$ has weight ≥ 2. Hence, $\rho_1(N) = O(1/N)$ at worst. Since $\zeta_N(1) = O(\log N)$, $\lim_{N\to\infty} \rho_1(N)\zeta_N(1) = 0$. Thus, the contribution of the last term on the right side of (3.41) approaches zero as $N \to \infty$. This finishes the proof of theorem 1. //

We will now show that values of single polylogarithms Li_k at roots of unity can be written in terms of single L-values, for $k > 1$.

Proposition 3. *Let $a \in \mathbf{Z}$, $a > 1$, $D \in \mathbf{Z}_+$, and $\zeta \in \mu_D$. Set $m = \operatorname{lcm}(D, \phi(D)$. Then $Li_a(\zeta)$ equals a linear combination*

$$\sum_{j=1}^{J} a_j L(a, \chi_j)$$

where each $a_j \in \mathbf{Q}(\zeta_m)$ and each χ_j is a Dirichlet character with conductor dividing D.

Proof: For $d|D$, we define

$$K_d := \{\text{Dirichlet characters}\, \chi : \operatorname{cond}(\chi)|d\}$$

and

$$\bar{U}_d := \left(\frac{\mathbf{Z}}{d\mathbf{Z}}\right)^* \qquad U_d := \pi_d^{-1}(\bar{U}_d) \qquad V_d := \frac{D}{d} U_d$$

where $\pi_d : \mathbf{Z} \to \mathbf{Z}/d\mathbf{Z} \supset \bar{U}_d$ is the canonical projection. Since each $\chi \in K_d$ is constant on cosets modulo $d\mathbf{Z}$, χ factors through π_d. So we can view χ as the induced map $\bar{U}_d \to \mu_d \cup \{0\}$. We need a preliminary lemma.

Lemma 1. *For any $d|D$, K_d is linearly independent, as a set of functions on $\bar{U}_d$.*

Proof: If $d = 1$, the lemma is clear, since in this case, K_d is a singleton set. Therefore, we can suppose $d > 1$. We define an Hermitian inner product on the set of all functions $\bar{U}_d \to \mathbf{C}$ by

$$< f, g >= \frac{1}{\phi(d)} \sum_{x \in \bar{U}_d} f(x)\overline{g(x)}$$

Obviously, $< \cdot, \cdot >$ is bilinear, and $< f, g >= \overline{< g, f >}$. If $f \neq 0$, then $< f, f >= \sum_{x \in \bar{U}_d} |f(x)|^2/\phi(d) > 0$. So $< \cdot, \cdot >$ is positive definite. Hence, $< \cdot, \cdot >$ is an Hermitian inner product.

We claim K_d is an orthonormal set under $< \cdot, \cdot >$. For $\chi, \psi \in K_d$,

$$< \chi, \psi >= \frac{1}{\phi(d)} \sum_{x \in \bar{U}_d} \chi(x)\overline{\psi(x)} = \frac{1}{\phi(d)} \sum_{x \in \bar{U}_d} \chi(xx')\overline{\psi(xx')}$$

for any $x' \in \bar{U}_d$, since $\bar{U}_d$ forms a group under the obvious multiplication. This equals

$$\frac{1}{\phi(d)} \sum_{x \in \bar{U}_d} \chi(x)\overline{\psi(x)}\chi(x')\overline{\psi(x')} = \frac{\chi(x')\overline{\psi(x')}}{\phi(d)} \sum_{x \in \bar{U}_d} \chi(x)\overline{\psi(x)}$$

which equals $\chi(x')\overline{\psi(x')} < \chi, \psi >$. Thus, if $\chi \neq \psi$, then $< \chi, \psi >= 0$ (since $d > 1$). If $\chi = \psi$, then $< \chi, \psi >=< \chi, \chi >= 1$. This shows that K_d is an orthonormal set under $< \cdot, \cdot >$, and is hence linearly independent. This proves the lemma. //

Now for $d|D$ and $\chi \in K_d$ define

$$f_\chi : \mathbf{Z} \to \mathbf{C} : m \mapsto \begin{cases} \chi(\frac{md}{D}), & \frac{D}{d}|m \\ 0, & \text{otherwise} \end{cases}$$

For $d|D$, define

$$L_d := \{f_\chi : \chi \in K_d\}$$

Also set

$$L := \bigcup_{d|D} L_d$$

Lemma 2. *The set L is linearly independent over* $\mathbf{C}$.

Proof: We first make the definition

$$M_d := \bigcup_{\substack{e>0 \\ d|e|D}} L_e$$

Now suppose we have a linear relation

$$\sum_{d|D} \sum_{f \in L_d} a_f f(n) = 0, \quad \forall n \in \mathbf{Z} \tag{2.43}$$

where each $a_f \in \mathbf{C}$. We will show that if $d|D$, and $a_f = 0$ for all $f \in M_d \backslash L_d$, then $a_f = 0$ for all $f \in L_d$. Note that this implies each $a_f = 0$, giving the lemma.

Let $d, e|D$. Suppose that for some $f \in L_e$, there is some integer $n \in V_d$ such that $f(n) \neq 0$. Since the support of f is contained in $\frac{D}{e}\mathbf{Z}$, we must have $\frac{D}{e}\mathbf{Z} \cap V_d \neq \phi$. Then, for some $m, u \in \mathbf{Z}$ with $(d, u) = 1$,

$$\frac{D}{e}m = \frac{D}{d}u$$

Hence, $dm = eu$. Now, the left side is divisible by d; hence, so is the right. Then $d|e$, since $(d, u) = 1$. Thus,

$$\{f \in L : f\big|_{V_d} \not\equiv 0\} \subset M_d$$

Using this, (2.43) implies

$$\sum_{f \in M_d} a_f f(n) = 0, \qquad \forall n \in V_d$$

By hypothesis, we know $a_f = 0$, for all $f \in M_d \backslash L_d$. Thus, we have

$$\sum_{f \in L_d} a_f f(n) = 0, \qquad \forall n \in V_d$$

which is equivalent to

$$\sum_{\chi \in K_d} a_{f_\chi} \chi(n) = 0, \qquad \forall n \in U_d$$

since each $f \in L_d$ equals f_χ for some $\chi \in K_d$. But by lemma 1, K_d is linearly independent as a set of functions on $\bar{U}_d$, and hence also linearly independent as a set of functions on U_d. So for each $\chi \in K_d$, $a_{f_\chi} = 0$. Then, for each $f \in L_d$, $a_f = 0$. Thus, for $f \in L$, all $a_f = 0$ in (2.43). This proves the lemma. //

To prove the proposition, we use the finite Fourier expansion (1.3) and (1.4) to write each $f \in L$ as a $\mathbf{Q}(\zeta_m)$-linear combination of the functions g_j, where

$$g_j : \mathbf{Z} \to \mathbf{C} : n \to \zeta_D^{jn}, \quad \text{for } j = 1, 2, \ldots, D$$

This is where we need $\phi(D)|m$ in the statement of the proposition, as each $f \in L$ takes values in $\mu_{\phi(D)} \cup \{0\}$. We have a system of linear equations with all coefficients in $\mathbf{Q}(\zeta_m)$, and each $f \in L$ is contained in $\text{span}_{\mathbf{C}}\{g_j\}$. The size of L is $\sum_{d|D} \phi(d) = D$ (see, for example, [3]). Set $W = \text{span}_{\mathbf{C}}(L)$. Then by lemma 2, $\dim W = D$, and L is a basis for W. We note that since $\text{span}_{\mathbf{C}}\{g_j\} \supset W$, $\text{span}_{\mathbf{C}}\{g_j\} = W$, whence the set $G = \{g_j\}$ is also a basis for W.

Now select an ordering $f^{(1)}, \ldots, f^{(D)}$ of L. Define a linear map $T : W \to W$ such that $T(g_j) = f^{(j)}$, for $j = 1, \ldots, D$. In the basis G, T has entries in $\mathbf{Q}(\zeta_m)$; as the image of T contains a basis, T is invertible, and the matrix of T^{-1} also has entries in $\mathbf{Q}(\zeta_m)$. Since $T^{-1}(f^{(j)}) = g_j$ for $j = 1, \ldots, D$, each g_j is a $\mathbf{Q}(\zeta_m)$-linear combination of the functions in L.

It follows from standard facts about Dirichlet series that for g_j as above and $\chi \in L$,

$$\sum_{n=1}^{\infty} \frac{g_j(n)}{n^a}, \qquad \sum_{n=1}^{\infty} \frac{f_\chi(n)}{n^a} \tag{2.44}$$

both converge absolutely for $a > 1$. Writing

$$g_j(n) = \sum_{f \in L} \alpha_f f(n)$$

for $\alpha_f \in \mathbf{Q}(\zeta_m)$, and substituting this into the first sum in (3.43) now proves the proposition, since we can change the order of summation. //

The proof of theorem 1 actually shows $L(\chi, \psi; a, b)$ can be written as a finite sum $\sum_j a_j l_j m_j$, where each $a_j \in \mathbf{Q}(\zeta_m)$, and each l_j and m_j are values of single polylogarithms at F-th roots of unity. Combining this fact with proposition 3, and using lemma 1 of theorem 1, we have

Corollary. *With hypotheses as in theorem 1, $L(\chi, \psi; a, b)$ can be written as a finite sum $\sum a_j l_j m_j$, where each l_j and m_j is either an L-series value or a value of Li_1 at an F-th root of unity, and each $a_j \in \mathbf{Q}(\zeta_m)$.*

We will finish this section by giving derivations verifying the identities (2.1) and (2.2). Let χ be the quadratic character of conductor 3. We will here find evaluations for

$$L\binom{1, \chi}{3, 1}, \qquad L\binom{\chi, 1}{1, 3}, \quad \text{and} \quad L\binom{\chi, 1}{2, 2}$$

in terms of single L-values and values of Li_1 at cube roots of unity. We will need the following partial fractions expansion (which is a special case of (2.8) and (2.9)):

$$\frac{1}{n(n-m)^3} = -\frac{1}{m^3 n} + \frac{1}{m(n-m)^3} - \frac{1}{m^2(n-m)^2} + \frac{1}{m^3(n-m)} \tag{2.45}$$

Set $\omega = \exp(2\pi i/3)$. We will make use of the formulas

$$Li_1(\omega) = -\frac{1}{2}\log 3 + \frac{i\pi}{6} \qquad Li_a(\omega) = \frac{\sqrt{-3}}{2} L_\chi(a) - \frac{3^{a-1} - 1}{2 \cdot 3^{a-1}} \zeta(a) \tag{2.46}$$

where $a \in \mathbf{Z}$, $a > 1$. The first of these is an application of (2.15). We find

$$L_N\binom{1,\chi}{3,1} = \sum_{n=1}^{N}\sum_{m=1}^{n-1}\frac{\chi(n)}{m^3 n} = \sum_{n=1}^{N}\sum_{m=1}^{n-1}\frac{\chi(n)}{(n-m)^3 n}$$

Applying the partial fractions expansion (2.45) to the last double sum, we obtain

$$\sum_{n=1}^{N}\sum_{m=1}^{n-1}\chi(n)\Big\{\frac{-1}{m^3 n} + \frac{1}{m(n-m)^3} - \frac{1}{m^2(n-m)^2} + \frac{1}{m^3(n-m)}\Big\}$$

$$= -L_N\binom{1,\chi}{3,1} + \sum_{m,n=1}^{N}\frac{\chi(m+n)}{mn^3} - \sum_{m,n=1}^{N}\frac{\chi(m+n)}{m^2n^2} + \sum_{m,n=1}^{N}\frac{\chi(m+n)}{m^3 n}$$

plus some error terms which approach zero as $N \to \infty$, by lemma 3 of theorem 1 . By noting that the second and last terms on the right-hand side of this expression are equal, we obtain

$$L_N\binom{1,\chi}{3,1} = \sum_{m,n=1}^{N}\frac{\chi(m+n)}{mn^3} - \frac{1}{2}\sum_{m,n=1}^{N}\frac{\chi(m+n)}{m^2n^2}$$

Using the finite Fourier expansion $\chi(n) = (\omega^n - \omega^{-n})/\sqrt{-3}$, the right-hand side of the last expression is

$$\frac{1}{\sqrt{-3}}\Big(Li_3^N(\omega)Li_1^N(\omega) - Li_3^N(\omega^{-1})Li_1^N(\omega^{-1}) - \frac{1}{2}Li_2^N(\omega)^2 + \frac{1}{2}Li_2^N(\omega^{-1})^2\Big)$$

$$= \frac{1}{\sqrt{-3}}\Big(2i\,\mathrm{Im}\,(Li_1^N(\omega)Li_3^N(\omega)) - i\,\mathrm{Im}\,(Li_2^N(\omega)^2)\Big)$$

Letting $N \to \infty$ and applying (2.46), we see this equals

$$\begin{aligned}
&\frac{1}{\sqrt{-3}}\Big\{2i\,\mathrm{Im}\,\Big((-\frac{1}{2}\log 3 + \frac{i\pi}{6})(\frac{\sqrt{-3}}{2}L_\chi(3) - \frac{4}{9}\zeta(3))\Big)\\
&- i\,\mathrm{Im}\,\Big((\frac{\sqrt{-3}}{2}L_\chi(2) - \frac{1}{3}\zeta(2))^2\Big)\Big\}\\
&= \frac{1}{\sqrt{-3}}\Big\{2i(-\frac{\sqrt{-3}}{4}\log 3 L_\chi(3) - \frac{2}{27}\pi\zeta(3)) - i(-\frac{1}{\sqrt{3}}L_\chi(2)\zeta(2))\Big\}\\
&= -\frac{1}{2}\log 3 L_\chi(3) - \frac{4\pi}{27\sqrt{3}}\zeta(3) + \frac{1}{3}L_\chi(2)\zeta(2)\\
&= -\frac{1}{2}\log 3 L_\chi(3) - \frac{4}{9}L_\chi(1)\zeta(3) + \frac{1}{3}L_\chi(2)\zeta(2)
\end{aligned}$$

since $L_\chi(1) = \pi/3\sqrt{3}$. Thus,

$$L\binom{1,\chi}{3,1} = -\frac{1}{2}\log 3 L_\chi(3) - \frac{4}{9}L_\chi(1)\zeta(3) + \frac{1}{3}L_\chi(2)\zeta(2)$$

Now, we can use the shuffle relation (2.20)

$$L\binom{\chi,1}{1,3} + L\binom{1,\chi}{3,1} + L_\chi(4) = L_\chi(1)\zeta(3)$$

since each term of this expression converges. This proves the evaluation

$$L\binom{\chi,1}{1,3} = \frac{1}{2}\log 3 L_\chi(3) + \frac{13}{9}L_\chi(1)\zeta(3) - \frac{1}{3}L_\chi(2)\zeta(2) - L_\chi(4)$$

verifying (2.1).

Similar methods show

$$K\binom{1,\chi}{1,3} = L\binom{1,\chi}{1,3} - L\binom{\chi,1}{2,2} - L\binom{\chi,1}{1,3} + L_\chi(4) \tag{2.47}$$

$$K\binom{1,\chi}{3,1} = L\binom{1,\chi}{1,3} + L_\chi(4) + L_\chi(1)\zeta(3) - L_\chi(2)\zeta(2) \tag{2.48}$$

Next we do a calculation:

$$\begin{aligned}\sum_{m,n=1}^{N} \frac{\chi(n-m)}{m^3 n} &= \sum_{0<m<n\le N} \frac{\chi(n-m)}{m^3 n} + \sum_{0<n<m\le N} \frac{\chi(n-m)}{m^3 n} \\ &= K_N\binom{1,\chi}{3,1} + \sum_{0<m<n\le N} \frac{\chi(m-n)}{mn^3} \\ &= K_N\binom{1,\chi}{3,1} - \sum_{0<m<n\le N} \frac{\chi(n-m)}{mn^3} \\ &= K_N\binom{1,\chi}{3,1} - K_N\binom{1,\chi}{1,3}\end{aligned}$$

By letting $N \to \infty$, we obtain

$$K\binom{1,\chi}{3,1} - K\binom{1,\chi}{1,3} = \sum_{m,n=1}^{\infty} \frac{\chi(n-m)}{m^3 n}$$

Subtracting (2.47) from (2.48) and solving for $L(\chi, 1; 2, 2)$, we obtain

$$L\binom{\chi,1}{2,2} = L_\chi(2)\zeta(2) - L_\chi(1)\zeta(3) - L\binom{\chi,1}{1,3} + \sum_{m,n=1}^{\infty} \frac{\chi(n-m)}{m^3 n} \tag{2.49}$$

We can calculate the last term as before, obtaining

$$\sum_{m,n=1}^{\infty} \frac{\chi(n-m)}{m^3 n} = \frac{1}{2}\log 3 L_\chi(3) - \frac{4}{9} L_\chi(1)\zeta(3)$$

Plugging this back into (2.49), we get

$$L\binom{\chi,1}{2,2} = L_\chi(4) + \frac{4}{3} L_\chi(2)\zeta(2) - \frac{26}{9} L_\chi(1)\zeta(3)$$

deriving (2.2).

3. Reductions of Some Triple L-Values

J. Borwein and R. Girgensohn[1] proved the analogue of Zagier's result (as stated in the introduction) for the triple zeta values. The technique of their proof does not seem to work for the general case of the triple L-values; however, it does work if the characters are fixed (although it becomes enormously complicated in all but the simplest cases). We obtain the following:

Theorem 2. *Suppose χ is one of the quadratic characters of conductor 3, 4, or 5, and D the conductor of χ. Let $m = \operatorname{lcm}(D, \phi(D))$. If $a, b, c \in \mathbf{Z}_+$ and $a + b + c \geq 4$ has the same parity as χ, then any of*

$$L\binom{\chi, 1, 1}{a, b, c}, \qquad L\binom{1, \chi, 1}{a, b, c}, \qquad L\binom{1, 1, \chi}{a, b, c} \tag{3.1}$$

(where in the first two cases we also require $c > 1$) lies in $R^2_{D,m}$.

Proof: If $c > 1$, then all three numbers in (3.1) converge, by Zhao's domain of absolute convergence. In the case of the third number in (3.1) where $c = 1$, we use the finite Fourier expansion (1.3) and (1.4) to expand in terms of triple polylogarithms evaluated at roots of unity. Since $\chi \neq 1$, the triple zeta value will not appear in this expansion; hence, by proposition 1, each term of this expansion will converge. Thus, the third number in (3.1) converges for any $c \in \mathbf{Z}_+$.

The strategy for the proof will be similar to that of the proof of theorem 1. We will use a decomposition equation, which is a relation derived from the same partial fractions method we used in the last section. In this section, the decomposition equation will be used to find linear combinations of triple sums which equal linear combinations of products of single and double sums, plus some error terms. In lemma 4, these error terms will be shown to differ from a linear combination of products of single and double sums by an expression which approaches zero as the upper limit tends to infinity. We will also use a permutation equation (which will here play a role analogous to that of the shuffle relations in the last section) to write linear combinations of triple sums as linear combinations of products of single and double sums.

However, the list of triple series which must be considered in order to construct block matrices analogous to (2.34) depends on the character. For this reason, the best we can do is to prove special cases. We will make use of several matrix identities to show particular block matrices have full rank when the weight has the appropriate parity. We will again use proposition 2 to prove these identities. At the end, we will once more (as was necessary in the last section) resolve the issues concerning what happens as we let the upper limit approach infinity.

For the purposes of this proof, we define partial sums of multiple polylogarithms

$$Li^N_{a_1,\ldots,a_d}(x_1,\ldots,x_d) := \sum_{0<n_1<\cdots<n_d} \frac{x_1^{n_1}\cdots x_d^{n_d}}{n_1^{a_1}\cdots n_d^{a_d}}$$

We make use of the following, which we will call the *permutation equation.*

Lemma 1. *We have the identity of partial sums of multiple polylogarithms*

$$\begin{aligned} &Li^N_{d,e,f}(x,y,z) + Li^N_{e,d,f}(y,x,z) + Li^N_{e,f,d}(y,z,x) \\ &\qquad = Li^N_d(x) Li^N_{e,f}(y,z) - Li^N_{d+e,f}(xy,z) - Li^N_{e,d+f}(y,xz) \end{aligned} \tag{3.2}$$

for $d, e, f \in \mathbf{Z}_+$.

Proof: We write

$$\begin{aligned} Li^N_d(x) Li^N_{e,f}(y,z) = &\sum_{\substack{0<n_1\leq N\\ 0<n_2<n_3\leq N}} \frac{x^{n_1}y^{n_2}z^{n_3}}{n_1^d n_2^e n_3^f} = \sum_{0<n_1<n_2<n_3\leq N} \frac{x^{n_1}y^{n_2}z^{n_3}}{n_1^d n_2^e n_3^f} \\ &+ \sum_{0<n_1<n_2\leq N} \frac{(xy)^{n_1}z^{n_2}}{n_1^{d+e} n_2^f} + \sum_{0<n_2<n_1<n_3\leq N} \frac{x^{n_1}y^{n_2}z^{n_3}}{n_1^d n_2^e n_3^f} \\ &+ \sum_{0<n_1<n_2\leq N} \frac{y^{n_1}(xz)^{n_2}}{n_1^e n_2^{d+f}} + \sum_{0<n_2<n_3<n_1\leq N} \frac{x^{n_1}y^{n_2}z^{n_3}}{n_1^d n_2^e n_3^f} \end{aligned}$$

$$\begin{aligned}= Li^N_{d,e,f}(x,y,z) + Li^N_{d+e,f}(xy,z) + Li^N_{e,d,f}(y,x,z)\\ + Li^N_{e,d+f}(y,xz) + Li^N_{e,f,d}(y,z,x)\end{aligned}$$

//

Note that lemma 1 gives an identity of multiple polylogarithms when the defining sum for each term converges. As in the case of the shuffle relations, we see here also that there are no error terms.

Next, we will derive a decomposition equation which will relate different types of triple sums, some of which are non-standard. We therefore begin by considering a triple Dirichlet series of the form

$$L(s_1,s_2,s_3) = \sum_{0<n_1<n_2<n_3} r(n_1,n_2,n_3) n_1^{-s_1} n_2^{-s_2} n_3^{-s_3}$$

where

$$r(n_1,n_2,n_3) = \chi(\alpha n_1 + \beta n_2 + \gamma n_3)$$

for some Dirichlet character χ and $\alpha,\beta,\gamma \in \mathbf{Z}$. We define the partial sums

$$L_N(s_1,s_2,s_3) = \sum_{0<n_1<n_2<n_3\le N} r(n_1,n_2,n_3) n_1^{-s_1} n_2^{-s_2} n_3^{-s_3}$$

To find relations among these numbers, we again use our partial fractions formulas (2.8) and (2.9):

$$\begin{aligned}L_N(a,b,c) &= \sum_{n_3=1}^{N}\sum_{n_2=1}^{n_3-1}\sum_{n_1=1}^{n_2-1} \frac{r(n_1,n_2,n_3)}{n_1^a n_2^b n_3^c}\\ &= \sum_{n_3=1}^{N}\sum_{n_2=1}^{n_3-1}\sum_{n_1=1}^{n_3-n_2-1} \frac{r(n_1,n_3-n_2,n_3)}{n_1^a (n_3-n_2)^b n_3^c}\\ &= \sum_{d+e=c+b} \{M^{c,b}_{d,e} \sum_{n_3=1}^{N}\sum_{n_2=1}^{n_3-1}\sum_{n_1=1}^{n_3-n_2-1} \frac{r(n_1,n_3-n_2,n_3)}{n_1^a n_3^d n_2^e}\\ &\quad + N^{c,b}_{d,e} \sum_{n_3=1}^{N}\sum_{n_2=1}^{n_3-1}\sum_{n_1=1}^{n_3-n_2-1} \frac{r(n_1,n_3-n_2,n_3)}{n_1^a n_2^d (n_3-n_2)^e}\} \qquad (3.3)\end{aligned}$$

where d,e are restricted to the positive integers. This equals

$$\begin{aligned}&\sum_{d+e=c+b} \{M^{c,b}_{d,e} \sum_{n_3=1}^{N}\sum_{n_2=1}^{n_3-1}\sum_{n_1=n_2+1}^{n_3-1} \frac{r(n_1-n_2,n_3-n_2,n_3)}{(n_1-n_2)^a n_3^d n_2^e}\\ &+ N^{c,b}_{d,e} \sum_{n_3=1}^{N}\sum_{n_2=1}^{n_3-1}\sum_{n_1=1}^{n_2-1} \frac{r(n_1,n_2,n_3)}{n_1^a (n_3-n_2)^d n_2^e}\}\\ &= \sum_{d+e=c+b} \{M^{c,b}_{d,e} \sum_{n_3=1}^{N}\sum_{n_1=1}^{n_3-1}\sum_{n_2=1}^{n_1-1} \frac{r(n_1-n_2,n_3-n_2,n_3)}{(n_1-n_2)^a n_3^d n_2^e}\\ &+ N^{c,b}_{d,e} \{\sum_{n_3=1}^{N}\sum_{n_2=1}^{N}\sum_{n_1=1}^{n_2-1} \frac{r(n_1,n_2,n_2+n_3)}{n_1^a n_3^d n_2^e} - e(a,e,d,N)\}\} \qquad (3.4)\end{aligned}$$

where

$$e(a,b,c,N) = \sum_{n_3=1}^{N}\sum_{n_2=N+1-n_3}^{N}\sum_{n_1=1}^{n_2-1} \frac{r(n_1,n_2,n_2+n_3)}{n_1^a n_2^b n_3^c} \qquad (3.5)$$

The triple sum in the last line of (3.4) can be written as a linear combination of products of single and double sums by applying the finite Fourier expansion (1.3) and (1.4). We also use this finite Fourier expansion to write $e(a,b,c,N)$ in terms of the numbers

$$e_N\binom{x,y,z}{a,b,c} = \sum_{n_3=1}^{N} \sum_{n_2=N+1-n_3}^{N} \sum_{n_1=1}^{n_2-1} \frac{x^{n_1}y^{n_2}z^{n_3}}{n_1^a n_2^b n_3^c} \tag{3.6}$$

where x, y, and z are roots of unity. We will show in lemma 4 that these error terms differ from a $\mathbf{Q}(\zeta_m)$-linear combination of products of single and double sums by an expression which approaches zero as $N \to \infty$. Then, assuming lemma 4, the following differs from (3.4) by a $\mathbf{Q}(\zeta_m)$-linear combination of products of single and double sums, plus an expression which approaches zero as $N \to \infty$:

$$\sum_{d+e=c+b} (-1)^a M_{d,e}^{c,b} \sum_{n_3=1}^{N} \sum_{n_1=1}^{n_3-1} \sum_{n_2=1}^{n_1-1} \frac{r(n_1-n_2, n_3-n_1, n_3)}{n_2^e (n_2-n_1)^a n_3^d}$$

$$= \sum_{d+e=c+b} (-1)^a M_{d,e}^{c,b} \sum_{f+g=e+a} \big\{ M_{f,g}^{e,a} \cdot$$

$$\sum_{n_3=1}^{N} \sum_{n_1=1}^{n_3-1} \sum_{n_2=1}^{n_1-1} \frac{r(n_1-n_2, n_3-n_2, n_3)}{n_2^f n_1^g n_3^d}$$

$$+ N_{f,g}^{e,a} \sum_{n_3=1}^{N} \sum_{n_1=1}^{n_3-1} \sum_{n_2=1}^{n_1-1} \frac{r(n_1-n_2, n_3-n_2, n_3)}{n_1^f (n_2-n_1)^g n_3^d} \big\}$$

$$= \sum_{d+e=c+b} (-1)^a M_{d,e}^{c,b} \sum_{f+g=e+a} \big\{ M_{f,g}^{e,a} \sum_{0<n_1<n_2<n_3\le N} \frac{r(n_2-n_1, n_3-n_1, n_3)}{n_1^f n_2^g n_3^d}$$

$$+ (-1)^g N_{f,g}^{e,a} \sum_{n_3=1}^{N} \sum_{n_1=1}^{n_3-1} \sum_{n_2=1}^{n_1-1} \frac{r(n_2, n_3-n_1+n_2, n_3)}{n_1^f n_2^g n_3^d} \big\}$$

$$= \sum_{d+e=c+b} (-1)^a M_{d,e}^{c,b} \sum_{f+g=e+a} \big\{ M_{f,g}^{e,a} \sum_{0<n_1<n_2<n_3\le N} \frac{r(n_2-n_1, n_3-n_1, n_3)}{n_1^f n_2^g n_3^d}$$

$$+ (-1)^g N_{f,g}^{e,a} \sum_{0<n_1<n_2<n_3\le N} \frac{r(n_1, n_3-n_2+n_1, n_3)}{n_1^g n_2^f n_3^d} \big\} \tag{3.7}$$

Then we can write $L_N(a,b,c)$ as (3.7) plus a sum of finitely many error terms of the form (3.6). We will call the equation expressing $L_N(a,b,c)$ in this manner the *decomposition equation*.

Set $w := a+b+c$. Let S_w be the set of positive integer points (d,e,f) such that $d+e+f = w$. Note $|S_w| = \binom{w-1}{2}$, which we will denote by v. Select an ordering

$$W : S_w \to \{1, \ldots, v\} \tag{3.8}$$

Given a $v \times v$ matrix M, we will denote $M_{W(a,b,c),W(d,e,f)}$ by

$$M\binom{a,b,c}{d,e,f}$$

Let P and Q be the permutation matrices defined by

$$P\binom{a,b,c}{d,e,f} = \delta_{a,e}\delta_{c,f} \qquad Q\binom{a,b,c}{d,e,f} = \delta_{b,d}\delta_{c,e} \tag{3.9}$$

where (a,b,c) and (d,e,f) are positive integer points with $a+b+c=d+e+f=w$ in the ordering W. Similarly define matrices A_1 and A_2 by

$$A_1\binom{a,b,c}{d,e,f} = (-1)^{b-1}\binom{b+c-f-1}{b-1}\binom{e-1}{a-1} \tag{3.10}$$

$$A_2\binom{a,b,c}{d,e,f} = (-1)^{b-1}\binom{e-1}{b-1}\binom{e-b}{c-f} \tag{3.11}$$

It is shown in [1] that

$$P^2 = Q^3 = I \qquad (PQ)^2 = I \tag{3.12}$$

We shall need the following.

Lemma 2.

$$A_1^3 = A_2^2 = I \qquad (A_1A_2)^2 = I \tag{3.13}$$

Proof:

$$A_2^2\binom{a,b,c}{d,e,f} = \sum(-1)^{b-1}\binom{h-1}{b-1}\binom{h-b}{c-i}(-1)^{h-1}\binom{e-1}{h-1}\binom{e-h}{i-f}$$

where the sum is over all positive integer points (g,h,i) such that $g+h+i=w$. An unindexed sigma will have this meaning in the computation of matrix identities in this proof. It is easy to see

$$\binom{h-1}{b-1}\binom{e-1}{h-1} = \binom{e-1}{b-1}\binom{e-b}{h-b}$$

by writing out both sides in terms of factorials. This technique will be used often (without comment) to show the matrix identities in this section. Now

$$A_2^2 = (-1)^{b-1}\binom{e-1}{b-1}\sum(-1)^{h-1}\binom{e-b}{h-b}\binom{h-b}{c-i}\binom{e-h}{i-f} \tag{3.14}$$

$$= (-1)^{b-1}\binom{e-1}{b-1}\sum(-1)^{h-1}\binom{i+e-b-c}{h+i-b-c}\binom{e-h}{i-f}\binom{e-b}{c-i}$$

which by Proposition 2 equals

$$(-1)^{b-1}\binom{e-1}{b-1}\sum_{i=1}^{w-2}(-1)^{i-b-c+1}\binom{0}{b+c-e-f}\binom{e-b}{c-i}$$

This last expression equals

$$= (-1)^{b-1}\binom{e-1}{b-1}\delta_{b+c,e+f}(-1)^{b+c}\sum_i(-1)^{i-1}\binom{e-b}{c-i}$$

$$= (-1)^{c-1}\binom{e-1}{b-1}\delta_{b+c,e+f}(-1)^{c-1}\delta_{b,e}$$

which equals the identity matrix.

To compute A_1^3, we first compute

$$A_1^2 = \sum(-1)^{b-1}\binom{b+c-i-1}{b-1}\binom{h-1}{a-1}(-1)^{h-1}\binom{h+i-f-1}{h-1}\binom{e-1}{g-1}$$

$$= (-1)^{b-1}\sum(-1)^{h-1}\binom{g+h-a-1}{b-1}\binom{d+e-a-g}{d+e-g-h}\cdot$$
$$\binom{d+e-g-1}{a-1}\binom{e-1}{g-1}$$

$$= (-1)^{b-1} \sum_g (-1)^{d+e-g-1} \binom{g-1}{c-f} \binom{d+e-g-1}{a-1} \binom{e-1}{g-1}$$

$$= (-1)^{b+d+e-1} \sum_g (-1)^{g-1} \binom{e-1}{c-f} \binom{e+f-c-1}{e-g} \binom{d+e-g-1}{a-1}$$

$$= (-1)^{b+d+e-1} \binom{e-1}{c-f} (-1)^{f-c} \binom{d-1}{b-1} = (-1)^{a-1} \binom{e-1}{c-f} \binom{d-1}{b-1} \tag{3.15}$$

Now

$$A_1^3 = \sum (-1)^{a-1} \binom{h-1}{c-i} \binom{g-1}{b-1} (-1)^{h-1} \binom{h+i-f-1}{h-1} \binom{e-1}{g-1}$$

$$= (-1)^{a-1} \binom{e-1}{b-1} \sum (-1)^{h-1} \binom{d+e-a-b}{g+h-a-b} \binom{d+e-g-1}{d+e-a-b} \binom{e-b}{g-b}$$

$$= (-1)^{a-1} \binom{e-1}{b-1} (-1)^{a+b} \sum_g (-1)^{g-1} \binom{d+e-g-1}{d+e-a-b} \binom{e-b}{g-b} \delta_{a+b,d+e}$$

$$= (-1)^{b-1} \binom{e-1}{b-1} (-1)^{b-1} \binom{e-1}{d-a} \delta_{a+b,d+e} = \binom{e-1}{b-1} \binom{d-1}{a-1} \delta_{a+b,d+e} = I$$

For the last identity,

$$A_1 A_2 = \sum (-1)^{b-1} \binom{b+c-i-1}{b-1} \binom{h-1}{a-1} (-1)^{h-1} \binom{e-1}{h-1} \binom{e-h}{i-f}$$

$$= (-1)^{b-1} \sum (-1)^{h-1} \binom{h-1}{a-1} \binom{e-1}{h-1} \binom{b+c-i-1}{b-1} \binom{e-h}{i-f}$$

$$= (-1)^{b-1} \binom{e-1}{a-1} \sum (-1)^{h-1} \binom{e-a}{h-a} \binom{e-h}{i-f} \binom{b+c-i-1}{b-1}$$

$$= (-1)^{b-1} \binom{e-1}{a-1} \sum (-1)^{h-1} \binom{e+f-a-i}{h-a} \binom{e-a}{i-f} \binom{b+c-i-1}{b-1}$$

The only non-zero term here has $i = e+f-a$ and $h = a$. Thus, we get

$$A_1 A_2 = (-1)^{a+b} \binom{e-1}{a-1} \binom{d-1}{b-1} \tag{3.16}$$

Hence,

$$(A_1 A_2)^2 = \sum (-1)^{a+b} \binom{h-1}{a-1} \binom{g-1}{b-1} (-1)^{g+h} \binom{e-1}{g-1} \binom{d-1}{h-1}$$

$$= (-1)^{a+b} \sum (-1)^{g+h} \binom{g-1}{b-1} \binom{e-1}{g-1} \binom{h-1}{a-1} \binom{d-1}{h-1}$$

$$= (-1)^{a+b} \binom{e-1}{b-1} \binom{d-1}{a-1} \sum (-1)^{g+h} \binom{e-b}{g-b} \binom{d-a}{h-a}$$

$$= (-1)^{a+b} \binom{e-1}{b-1} \binom{d-1}{a-1} \delta_{e,b} \delta_{a,d} = I$$

finishing the proof of the lemma. //

The following lemma plays a pivotal role:

Lemma 3. *Let $N_1 = QPA_1$, $N_2 = QPA_2$. Then*

$$N_2^2 = I \qquad (N_1N_2)^2 = I \qquad N_2N_1N_2 = (-1)^{w-1}N_1^2 \tag{3.17}$$

Proof: One computes $QP = \delta_{b,e}\delta_{a,f}$ and

$$N_2 = (-1)^{b-1}\binom{e-1}{b-1}\binom{e-b}{a-f} \tag{3.18}$$

Then

$$N_2^2 = \sum(-1)^{b-1}\binom{h-1}{b-1}\binom{h-b}{a-i}(-1)^{h-1}\binom{e-1}{h-1}\binom{e-h}{g-f}$$

where again an unindexed sigma indicates summation over all positive integer points (g,h,i), with $g+h+i = w$. This equals

$$\begin{aligned}
&(-1)^{b-1}\sum(-1)^{h-1}\binom{h-1}{b-1}\binom{e-1}{h-1}\binom{h-b}{a-i}\binom{e-h}{d+e-h-i}\\
&= (-1)^{b-1}\binom{e-1}{b-1}\sum(-1)^{h-1}\binom{e-b}{h-b}\binom{h-b}{a-i}\binom{e-h}{d+e-h-i}\\
&= (-1)^{b-1}\binom{e-1}{b-1}\sum(-1)^{h-1}\binom{i+e-a-b}{e-h}\binom{e-h}{d+e-h-i}\binom{e-b}{a-i}\\
&= (-1)^{b-1}\binom{e-1}{b-1}\sum(-1)^{h-1}\binom{d+e-a-b}{h+i-a-b}\binom{e-b}{a-i}\binom{i+e-a-b}{d+e-a-b}\\
&= (-1)^{a-1}\binom{e-1}{b-1}\sum_i(-1)^{i-1}\binom{e-b}{a-i}\binom{i+e-a-b}{d+e-a-b}\delta_{a+b,d+e}\\
&= \binom{e-1}{b-1}\delta_{b,e}\delta_{a+b,d+e} = I
\end{aligned}$$

We find

$$N_1 = (-1)^{b-1}\binom{a+b-f-1}{b-1}\binom{e-1}{c-1} \tag{3.19}$$

so that

$$\begin{aligned}
N_1N_2 &= \sum(-1)^{b-1}\binom{a+b-i-1}{b-1}\binom{h-1}{c-1}(-1)^{h-1}\binom{e-1}{h-1}\binom{e-h}{g-f}\\
&= (-1)^{b-1}\binom{e-1}{c-1}\sum(-1)^{h-1}\binom{e-c}{e-h}\binom{e-h}{i-d}\binom{a+b-i-1}{b-1}\\
&= (-1)^{b-1}\binom{e-1}{c-1}\sum(-1)^{h-1}\binom{d+e-c-i}{h-c}\binom{e-c}{i-d}\binom{a+b-i-1}{b-1}\\
&= (-1)^{b+c}\binom{e-1}{c-1}\sum_i\delta_{i,d+e-c}\binom{e-c}{i-d}\binom{a+b-i-1}{b-1}\\
&= (-1)^{b+c}\binom{e-1}{c-1}\binom{f-1}{b-1}
\end{aligned} \tag{3.20}$$

Now

$$\begin{aligned}
(N_1N_2)^2 &= \sum(-1)^{b+c}\binom{h-1}{c-1}\binom{i-1}{b-1}(-1)^{h+i}\binom{e-1}{i-1}\binom{f-1}{h-1}\\
&= (-1)^{b+c}\sum(-1)^{h+i}\binom{h-1}{c-1}\binom{f-1}{h-1}\binom{i-1}{b-1}\binom{e-1}{i-1}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{b+c}\binom{f-1}{c-1}\binom{e-1}{b-1}\sum(-1)^{h+i}\binom{f-c}{h-c}\binom{e-b}{i-b}\\
&= \binom{f-1}{c-1}\binom{e-1}{b-1}\delta_{c,f}\delta_{b,e} = I.
\end{aligned}
$$

For the last identity (using the expression we found above for N_1N_2),

$$
\begin{aligned}
N_2N_1N_2 &= \sum(-1)^{b-1}\binom{h-1}{b-1}\binom{h-b}{a-i}(-1)^{h+i}\binom{e-1}{i-1}\binom{f-1}{h-1}\\
&= (-1)^{b-1}\binom{f-1}{b-1}\sum(-1)^{h+i}\binom{f-b}{h-b}\binom{h-b}{a-i}\binom{e-1}{i-1}\\
&= (-1)^{b-1}\binom{f-1}{b-1}\sum(-1)^{h+i}\binom{i+f-a-b}{h+i-a-b}\binom{f-b}{a-i}\binom{e-1}{i-1}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{b-1}\binom{f-1}{b-1}\sum_{i}(-1)^{a+b}\delta_{i,a+b-f}\binom{f-b}{a-i}\binom{e-1}{i-1}\\
&= (-1)^{a-1}\binom{f-1}{b-1}\binom{e-1}{d-c} \qquad (3.21)
\end{aligned}
$$

Also

$$
\begin{aligned}
N_1^2 &= \sum(-1)^{b-1}\binom{a+b-i-1}{b-1}\binom{h-1}{c-1}(-1)^{h-1}\binom{g+h-f-1}{h-1}\binom{e-1}{i-1}\\
&= (-1)^{b-1}\sum(-1)^{h-1}\binom{d+e-c-i}{h-c}\binom{d+e-i-1}{c-1}\cdot\\
&\qquad\qquad\binom{a+b-i-1}{b-1}\binom{e-1}{i-1}\\
&= (-1)^{b+c}\sum_{i}\delta_{i,d+e-c}\binom{d+e-i-1}{c-1}\binom{a+b-i-1}{b-1}\binom{e-1}{i-1}\\
&= (-1)^{b+c}\binom{f-1}{b-1}\binom{e-1}{d-c} \qquad (3.22)
\end{aligned}
$$

as desired. //

Given a $v \times v$ matrix M, we denote its minimal polynomial by $f_M(X)$. Using the above lemmas, one finds the following

$$
\begin{aligned}
f_{A_1+A_2}(X) &\quad \text{divides} \quad X(X-2)(X+1) \qquad (3.23)\\
f_{A_1-A_2}(X) &\quad \text{divides} \quad X(X-2)(X+1) \qquad (3.24)\\
f_{N_1+N_2}(X) &\quad \text{divides} \quad X(X+(-1)^{w}2)(X+(-1)^{w-1}) \qquad (3.25)
\end{aligned}
$$

by plugging the indicated matrices into the given polynomials, and obtaining the zero matrix.

In the next part of the proof, we will use the permutation equation and the decomposition equation to construct $\mathbf{Z}$-linear combinations of the partial sums of certain triple Dirichlet series which equal $\mathbf{Q}(\zeta_m)$-linear combinations of products of single and double sums. The triple Dirichlet series involved will be different for each of the three quadratic characters we are considering. In each case, the coefficients of these linear combinations will form a large block matrix, which we will show has full rank in the case that $\chi(-1) = (-1)^w$. The upper half of this matrix will consist of the coefficients of the linear combinations arising from the use of the permutation equation, and will involve the smaller blocks P, Q. The lower half of the matrix will similarly consist of the coefficients of the linear combinations arising from the use of the decomposition equation, and will involve the smaller blocks A_1, A_2.

In our row-reductions of matrices which we shall construct, we will use some shorthand language to describe the operations we will perform. Consider a block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.26}$$

where each block is a square matrix, and each block has the same dimension. Also suppose A is invertible. Note that in row-reducing this matrix, we may multiple each entry of a row by a fixed square matrix on the left, since this corresponds to row operations; we may likewise multiply each matrix in a column by a fixed square matrix on the right, as this corresponds to column operations. We can multiply the first row of (3.26) by A^{-1} on the left, and then subtract from the second row C times the first row. Multiplying the first row now by A results in the block matrix

$$\begin{pmatrix} A & B \\ & D - CA^{-1}B \end{pmatrix} \tag{3.27}$$

We will call this procedure "using row 1 to row-reduce row 2."

First suppose $D = 3$. At each positive integer point (s, t, u) such that

$$s + t + u = a + b + c = w$$

with w odd, we consider the 13 triple series

$$L\begin{pmatrix}\chi, 1, 1 \\ s, t, u\end{pmatrix} \qquad L\begin{pmatrix}1, \chi, 1 \\ s, t, u\end{pmatrix} \qquad L\begin{pmatrix}1, 1, \chi \\ s, t, u\end{pmatrix} \qquad \sum \frac{\chi(n_2 - n_1)}{n_1^s n_2^t n_3^u}$$
$$\sum \frac{\chi(n_3 - n_2)}{n_1^s n_2^t n_3^u} \qquad \sum \frac{\chi(n_3 - n_1)}{n_1^s n_2^t n_3^u} \qquad \sum \frac{\chi(n_1 + n_2)}{n_1^s n_2^t n_3^u}$$
$$\sum \frac{\chi(n_2 + n_3)}{n_1^s n_2^t n_3^u} \qquad \sum \frac{\chi(n_1 + n_3)}{n_1^s n_2^t n_3^u} \qquad \sum \frac{\chi(n_1 + n_2 - n_3)}{n_1^s n_2^t n_3^u} \tag{3.28}$$
$$\sum \frac{\chi(n_1 - n_2 + n_3)}{n_1^s n_2^t n_3^u} \qquad \sum \frac{\chi(-n_1 + n_2 + n_3)}{n_1^s n_2^t n_3^u} \qquad \sum \frac{\chi(n_1 + n_2 + n_3)}{n_1^s n_2^t n_3^u}$$

where the sums have the restriction $0 < n_1 < n_2 < n_3 \leq N$. We denote the series in (4.28) by $S_1(p)$, $S_2(p)$, ..., $S_{13}(p)$, in the order listed, where $p = (s, t, u)$.

We will first use the permutation equation (3.2) to construct $\mathbf{Z}$-linear combinations of numbers in (3.28) which equal $\mathbf{Q}(\zeta_m)$-linear combinations of products of single and double sums. We do this by using the finite Fourier expansion (1.3) and (1.4) to write each $S_j(p)$ (where p is a positive integer point of weight w) as a sum of partial triple polylogarithms evaluated at cube roots of unity. We then apply the permutation equation to each term of this expansion, and rewrite the resulting partial triple polylogarithm sums in terms of the values of S_1, S_2, ..., S_{13}, and values of partial single and double polylogarithms at cube roots of unity. To illustrate, we first need the finite Fourier expansion (1.3) for χ

$$\chi(n) = \frac{1}{\sqrt{-3}}\left(\zeta_3^n - \zeta_3^{-n}\right)$$

We then use this expansion to write

$$S_4(s, t, u) = \frac{1}{\sqrt{-3}}\left(Li_{s,t,u}(\zeta_3^{-1}, \zeta_3, 1) - Li_{s,t,u}(\zeta_3, \zeta_3^{-1}, 1)\right)$$

Applying the permutation equation, we see

$$S_4(s, t, u) - S_4(t, s, u) - S_6(t, u, s) = \frac{1}{\sqrt{-3}}\big(Li_s^N(\zeta_3^{-1})Li_{t,u}^N(\zeta_3, 1)$$

$$-Li_s^N(\zeta_3)Li_{t,u}^N(\zeta_3^{-1},1)\big) - \frac{1}{\sqrt{-3}}\big(Li_{t,s+u}^N(\zeta_3,\zeta_3^{-1}) - Li_{t,s+u}^N(\zeta_3^{-1},\zeta_3)\big)$$

Thus, we have a linear combination of S_4 and S_6 sums which can be written as a linear combination of products of partial single and double polylogarithms evaluated at cube roots of unity. We can proceed in this manner for all of the sums listed in (3.28), as follows.

Given any number $S_j(p)$ in this list, we write the finite Fourier expansion

$$S_j(p) = \sum_k \alpha_k Li_{s,t,u}^N(\zeta_{1,k},\zeta_{2,k},\zeta_{3,k})$$

for roots of unity $\zeta_{l,k}$. We then plug each partial triple polylogarithm term which appears in this summand into the first term on the left-hand side of the permutation equation (3.2). By adding together the resulting equations, and rewriting the triple sums which appear in this sum of equations in terms of numbers in (3.28), we obtain a **Z**-linear combination of numbers in (3.28) which equals a $\mathbf{Q}(\zeta_m)$-linear combination of products of partial single and double polylogaritms evaluated at cube roots of unity. It turns out that the triple sums which are related by this method are all contained in the list (3.28).

We order the numbers (3.28) as follows

$$\begin{gathered} S_1(W^{-1}(1)),\ldots,S_1(W^{-1}(v)),S_2(W^{-1}(1)),\ldots, \\ S_2(W^{-1}(v)),\ldots,S_{13}(W^{-1}(1))\ldots,S_{13}(W^{-1}(v)) \end{gathered} \tag{3.29}$$

where W is the ordering (3.8). We form a *permutation matrix* $\tilde{P}$ by declaring $\tilde{P}_{ij}$ to be the coefficient of the j-th number in (3.29) in the relation obtained by applying the permutation equation (3.2) to the i-th number in (3.29) (as described above). In block matrix form, $\tilde{P}$ has entries involving the $v \times v$ block matrices P, Q defined in (3.9), and equals

$$\begin{pmatrix}
I & P & Q & & & & & & & & & & \\
R & I & & & & & & & & & & & \\
 & Q & N & & & & & & & & & & \\
 & & & M & & -Q & & & & & & & \\
 & & & Q & I & P & & & & & & & \\
 & & & & S & I & & & & & & & \\
 & & & & & & N & & Q & & & & \\
 & & & & & & Q & I & P & & & & \\
 & & & & & & & R & I & & & & \\
 & & & & & & & & & N & Q & & \\
 & & & & & & & & & & I & R & \\
 & & & & & & & & & Q & P & I & \\
 & & & & & & & & & & & & T
\end{pmatrix} \tag{3.30}$$

where

$$R = P + Q \qquad N = I + P \qquad M = I - P$$

$$S = P - Q \qquad T = I + P + Q$$

One can similarly apply the decomposition equation to each of the numbers in (3.29) by setting $L_N(a,b,c)$ in (3.3) equal to $S_j(p)$. This produces **Z**-linear combinations of the numbers in (3.29) which equal $\mathbf{Q}(\zeta_m)$-linear combinations of partial sums of single and double polylogarithms evaluated at cube roots of unity, plus error terms of the form (3.6). We define a *decomposition matrix* $\tilde{D}$ by declaring $\tilde{D}_{ij}$ to be the coefficient of the j-th number in (3.29) in the relation obtained by applying the decomposition equation (3.7) to the i-th number in (3.29). One can use (3.7) to calculate $\tilde{D}$ in block matrix form, resulting in entries involving A_1 and A_2. One obtains $\tilde{D} =$

$$\begin{pmatrix}
C & & & A_1 & & & & & & & & & \\
& I & & & & A_1 & & & & & A_2 & & \\
& & E & & & & & & & & & & \\
& & & I & D_1 & & & & & & & & \\
A_1 & & & A_2 & I & & & & & & & & \\
& & & & & C & & & & & A_1 & & \\
& & & & & & I & & & -A_2 & & & A_1 \\
& & & & & & & I & -A_1 & & & -A_2 & \\
& & & & & & & & C & & & A_1 & \\
& & & & & & D_2 & & & I & & & \\
& D_1 & & & & & & & & & I & & \\
& & & & & & & -D_1 & & & & I & \\
& & & & & & & & & A_1 & & & C_2
\end{pmatrix} \tag{3.31}$$

where

$$E = I + A_1 + A_2 \qquad D_1 = A_1 + A_2 \qquad D_2 = A_1 - A_2$$

$$C = I + A_2 \qquad C_2 = I - A_2$$

It is again important that all triple sums which can appear in these relations are contained in the list (3.28); in fact, the list was constructed so that this would be the case.

We now adjoin the decomposition matrix to the bottom of the permutation matrix. We will now perform a row- and column-reduction procedure which will give us an upper triangular square block matrix in the lower half, with each diagonal block invertible. I.e., after we are finished, the jth row of the lower half of the resulting matrix will have zero matrices in each of the first $j-1$ blocks, and some invertible matrix in the jth block. We have this already in rows 2 and 4. The following is a list of row and column operations which achieve our goal.

Row 1: Switch decomposition row 1 and permutation row 1.

Row 3: Subtract from row 3 $2Q^2$ times the row resulting from subtracting P times decomposition row 2 and A_1^2 times decompostion row 5 from decomposition row 1. This gives the diagonal entry $A_1 + A_2 - I$, which is invertible by (3.23) since 1 is not an eigenvalue of $A_1 + A_2$.

Row 5: Construct two new rows by row reducing permutation row 4 using decomposition row 4, and decomposition row 4 using permutation row 5. Multiply the difference of the two rows by Q, and put this in place of decomposition row 5. We get the diagonal entry $QPD_1 - I = N_1 + N_2 - I$, which is invertible by (3.25) since 1 is not an eigenvalue of $N_1 + N_2$ when w is odd.

Row 7: Switch permutation row 7 and decomposition row 7.

Row 8: Add permutation row 7 to decomposition row 8. From this subtract decomposition row 7 and permutation row 9. This gives the diagonal entry $I - P - Q$, which is invertible (since it turns out that 1 is not an eigenvalue of $P + Q$).

Row 9: Here we use a column operation. Subtract D_2 times column 9 from column 2.

Row 6: Switch decomposition row 8 and permutation row 9. Next multiply decomposition row 8 by zero. Subtract column 8 from column 6. We get the diagonal entry $I + A_2 - A_1$, which is invertible by (3.24). Now switch decomposition row 8 and permutation row 9 to get a lower triangular matrix in the bottom half such that each diagonal entry is an invertible, square matrix.

This shows our original block matrix (obtained by adjoining $\tilde{D}$ to the bottom of $\tilde{P}$) indeed has full rank, in the case in which w is odd. Thus, this linear system of equations can be solved, giving an expression in terms of single and double sums and error terms for each of the numbers (3.29).

In the case $D = 4$, an exactly analogous method replaces (3.28) with a list of 28 series. Thus, we get a permutation matrix and a decomposition matrix, each of which is a 28×28 block matrix. A similar row-reduction shows that when w is odd, the block matrix formed by adjoining these two matrices top to bottom has full rank. The step in which the parity of the weight w is used again requires the invertibility of $N_1 + N_2 - I$, which holds for odd w by (3.25).

In the case $D = 5$, we must consider a list of 37 series. This time each of our two block matrices are 37×37, and a similar row-reduction carries through in the case that w is even. The step in which the parity of w is used requires the invertibility of $N_1 + N_2 + I$, which holds for even w, also by (3.25).

In each of the three cases, solving a large block matrix writes each considered sum as a $\mathbf{Q}(\zeta_m)$-linear combination of partial sums of single and double polylogarithm sums evaluated at third, fourth, or fifth roots of unity, plus possibly some error terms. These error terms take the form

$$e_N\binom{x,y,z}{a,b,c} = \sum_{n_3=1}^{N} \sum_{n_2=N+1-n_3}^{N} \sum_{n_1=1}^{n_2-1} \frac{x^{n_1}y^{n_2}z^{n_3}}{n_1^a n_2^b n_3^c}$$

as given in (3.6). We also define

$$e_N\binom{x,y}{a,b} = \sum_{n=1}^{N} \sum_{m=N+1-n}^{N} \frac{x^m y^n}{m^a n^b} \tag{3.32}$$

where x, y, z are roots of unity and $a, b \in \mathbf{Z}_+$. We now find linear combinations of single and double sums which differ from each error term (3.6) by an expression which approaches zero as $N \to \infty$.

Lemma 4. *Let x, y, z be roots of unity, and $a, b, c \in \mathbf{Z}_+$ with $a + b + c \geq 4$. If $b + c \geq 3$, then*

$$\lim_{N\to\infty} e_N\binom{x,y,z}{a,b,c} = 0 \tag{3.33}$$

If $b = c = 1$, then

$$\lim_{N\to\infty} e_N\binom{x,y,z}{a,1,1} - Li_a^N(x)\Big\{Li_1^N(y)Li_1^N(z) - Li_{1,1}^N(\frac{y}{z}, z) - Li_{1,1}^N(\frac{z}{y}, y)\Big\} = 0 \tag{3.34}$$

Proof: For $N \geq 1$, $e_N(x, y; 1, 1) = \sum x^m y^n/mn$, where the sum is over ordered pairs $(m, n) \in \mathbf{Z}_+ \times \mathbf{Z}_+ \subset \mathbf{R} \times \mathbf{R}$ lying in the triangular region defined by $m, n \leq N$, $m + n > N$. By subtracting two such sums, we see

$$e_{N+1}\binom{x,y}{1,1} - e_N\binom{x,y}{1,1} = \frac{x^{N+1}}{N+1}\sum_{n=1}^{N}\frac{y^n}{n} + \frac{y^{N+1}}{N+1}\sum_{m=1}^{N}\frac{x^m}{m}$$

$$+\frac{(xy)^{N+1}}{(N+1)^2} - \sum_{n=1}^{N}\frac{x^n y^{N+1-n}}{n(N+1-n)}$$

By applying partial fractions, the last term on the right side equals

$$\frac{1}{N+1}\Big\{y^{N+1}\sum_{n=1}^{N}\frac{(x/y)^n}{n} + x^{N+1}\sum_{n=1}^{N}\frac{(y/x)^n}{n}\Big\}$$

In terms of values of double polylogarithms, we have

$$\begin{aligned}
e_N\binom{x,y}{1,1} &= e_1\binom{x,y}{1,1} + \sum_{m=1}^{N-1}(e_{m+1}\binom{x,y}{1,1} - e_m\binom{x,y}{1,1}) \\
&= Li_2^N(xy) + Li_{1,1}^N(y,x) + Li_{1,1}^N(x,y) - Li_{1,1}^N(x/y,y) - Li_{1,1}^N(y/x,x) \\
&= Li_2^N(xy) + Li_1^N(x)Li_1^N(y) - Li_2^N(xy) - Li_{1,1}^N(x/y,y) - Li_{1,1}^N(y/x,x) \\
&= Li_1^N(x)Li_1^N(y) - Li_{1,1}^N(x/y,y) - Li_{1,1}^N(y/x,x)
\end{aligned} \tag{3.35}$$

by using the shuffle relation

$$Li_{d,e}^N(x',y') + Li_{e,d}^N(y',x') + Li_{d+e}(x'y') = Li_d(x')Li_e(y')$$

for $d, e \in \mathbf{Z}_+$ and $x', y' \in \mathbf{C}$. On the other hand, duplicating the method of the proof of lemma 3 of theorem 1 shows

$$\lim_{N\to\infty} e_N\binom{x, y}{c, d} = 0 \tag{3.36}$$

if $c, d \in \mathbf{Z}_+$, $c + d \geq 3$. Note

$$Li_a^N(x)e_N\binom{y, z}{b, c} - e_N\binom{x, y, z}{a, b, c} = \sum_{n_3=1}^{N}\sum_{n_2=N+1-n_3}^{N}\sum_{n_1=n_2}^{N}\frac{x^{n_1}y^{n_2}z^{n_3}}{n_1^a n_2^b n_3^c}$$

Since $w \geq 4$, we can assume one of a, b, c is > 1. If $a > 1$, then

$$Li_a^N(x)e_N\binom{y, z}{b, c} - e_N\binom{x, y, z}{a, b, c} = O\Big\{\sum_{n_3=1}^{N}\sum_{n_2=N+1-n_3}^{N}(\frac{1}{N} - \frac{1}{n_2})\frac{y^{n_2}z^{n_3}}{n_2 n_3}\Big\}$$

$$= O\Big\{\frac{1}{N}e_N\binom{y, z}{1, 1} - e_N\binom{y, z}{2, 1}\Big\}$$

Note $e_N(y, z; 1, 1)$ is majorized by $\zeta_N(1)^2 = O(\log^2 N)$, and by (3.36), $e_N(y, z; 2, 1) \to 0$ as $N \to \infty$. Thus, the last expression approaches zero as $N \to \infty$. By applying (3.35), this proves the lemma in the case $a > 1$.

If $c > 1$ (by symmetry, this also covers the case in which $b > 1$),

$$e_N\binom{x, y, z}{a, b, c} = O\{\sum_{n_3=1}^{N}\sum_{n_2=N+1-n_3}^{N}(\log N - \log n_2)\frac{y^{n_2}z^{n_3}}{n_2^b n_3^c}\}$$

$$= O\sum_{n_3=1}^{N}\log^2 N\frac{z^{n_3}}{n_3^c} = O(\frac{\log^2 N}{N})$$

which also approaches zero as $N \to \infty$. Thus,

$$\lim_{N\to\infty}\Big\{e_N\binom{x, y, z}{a, b, c} - Li_a^N(x)e_N\binom{y, z}{b, c}\Big\} = 0 \tag{3.37}$$

Since here $b + c \geq 3$, $e_N(y, z; b, c) \to 0$ as $N \to \infty$. This shows

$$\lim_{N\to\infty} e_N\binom{x, y, z}{a, b, c} = 0$$

in the cases $b > 1$ and $c > 1$. We have proven the lemma. //

The expressions obtained by solving the permutation/decomposition matrix are $\mathbf{Q}(\zeta_m)$-linear combinations of products of partial single and double polylogarithms evaluated at roots of unity, plus possibly some error terms of the form (3.6). We can ignore the error terms with $b + c \geq 3$, as these approach zero as $N \to \infty$. We replace each error term for which $b = c = 1$ with the expression

$$Li_a^N(x)\Big\{Li_1^N(y)Li_1^N(z) - Li_{1,1}^N(\frac{y}{z}, z) - Li_{1,1}^N(\frac{z}{y}, y)\Big\}$$

which we can do, since the difference approaches zero as $N \to \infty$. This gives evaluations in terms of partial single and double polylogarithm sums evaluated at roots of unity. As in the proof of theorem 1, the resulting reduction may involve sums which could diverge as $N \to \infty$ (as far as we know). Those which may appear are $\zeta_N(1)$, $Li_{a,b}^N(\eta, 1)$ with $a > 1$ and $\eta \neq 1$ a root of unity, and $\zeta_N(1, 1)$. We must show that summands which contain one of these as a factor approach a number lying in $R_{D,m}^2$ as $N \to \infty$. To do this, we essentially duplicate an argument of [1]. Define

$$\zeta_N(1, 1) = \sum_{0<m<n\leq N}\frac{1}{mn} = Li_{1,1}^N(1, 1) \tag{3.38}$$

Then the shuffle relation says

$$2\zeta_N(1,1) + \zeta_N(2) = \zeta_N(1)^2 \tag{3.39}$$

So $\zeta_N(1,1) = O(\log^2 N)$. The shuffle relation of double polylogarithms says

$$Li^N_{c,d}(x', y') + Li^N_{d,c}(y', x') + Li^N_{c+d}(x'y') = Li^N_c(x')Li^N_d(y')$$

for $c, d \in \mathbf{Z}_+$ and $x', y' \in \mathbf{C}$. Thus,

$$Li^N_{a,1}(x, 1) + Li^N_{1,a}(1, x) + Li^N_{a+1}(x) = Li^N_a(x)\zeta_N(1)$$

so that

$$Li^N_{a,1}(x, 1) = O(\log N) \tag{3.40}$$

when either $a > 1$ or $a = 1$ and $x \neq 1$ is a root of unity. Let

$$S_N(a, b, c) = \sum_{0<n_1<n_2<n_3\leq N} \frac{r(n1, n2, n3)}{n_1^a n_2^b n_3^c}$$

be any sum of which our method produces an evaluation, and which converges either by Zhao's domain of absolute convergence, or by using the finite Fourier expansion and applying proposition 1. Thus, S_N may be any convergent sum in (3.28), or any convergent sum in analogous lists which arises for $D = 4$ or for $D = 5$. We write such an evaluation in the form

$$S_N(a, b, c) = \rho_0(N) + \rho_1(N)\zeta_N(1) + \rho_2(N)\zeta_N(1)^2 \tag{3.41}$$

where each $\rho_j(N)$ is a $\mathbf{Q}(\zeta_m)$-linear combination of products of partial single and double polylogarithms evaluated at roots of unity, each summand of which converges either by Zhao's domain of absolute convergence, or by using the finite Fourier expansion, and applying proposition 1. Since our evaluation techniques all preserve weight, there can be no $\zeta_N(1)^3$ term. Dividing both sides by $\zeta_N(1)^2$, we see $\rho_2(N) \to 0$ as $N \to \infty$, since the other terms do. Now $\rho_2(N)$ is a linear combination of partial single polylogarithms evaluated at roots of unity, each of weight at least two. Thus, $\rho_2(N)$ approaches zero at worst as $1/N$ as $N \to \infty$. Therefore, $\lim_{N\to\infty} \rho_2(N)\zeta_N(1)^2 = 0$. Henceforth, we ignore the term $\rho_2(N)\zeta_N(1)^2$. Multiplying by $\zeta_N(1)$ shows $\rho_1(N)$ also approaches zero; the latter consists of a $\mathbf{Q}(\zeta_m)$-linear combination of terms, each of which is either a product of no more than two partial single polylogarithms evaluated at roots of unity, or a partial double polylogarithm evaluated at roots of unity. Since $w \geq 4$, each summand of $\rho_1(N)$ has weight ≥ 3. It follows that each of these summands approaches zero no slower than $\log N/N$. Then $\rho_1(N)\zeta_N(1) = O(\log^2 N/N)$. Hence, $\lim_{N\to\infty} \rho_1(N)\zeta_N(1) = 0$. Thus, letting $N \to \infty$ in (3.41) gives an evaluation of S_N which is correct, and which lies in $R^2_{D,m}$. We have proven theorem 2. //

4. Series for Calculation

In order to simplify our formulas, we redefine the Bernoulli numbers β_n

$$\sum_{n=1}^{\infty} e^{-nt} = \sum_{n=-1}^{\infty} \beta_n t^n, \qquad \Re(t) > 0, \quad |t| < 2\pi \tag{4.1}$$

and the Bernoulli numbers $\beta_{n,\chi}$ associated to a character χ

$$\sum_{n=1}^{\infty} \chi(n) e^{-nt} = \sum_{n=0}^{\infty} \beta_{n,\chi} t^n, \qquad \Re(t) > 0, \quad |t| < \frac{2\pi}{D} \tag{4.2}$$

where D is the conductor of χ. Expanding in residue classes gives the formula

$$\beta_{n,\chi} = \sum_{a=1}^{D-1} \chi(a) \sum_{m=0}^{n+1} \frac{(D-a)^m}{m!} D^{n-m} \beta_{n-m} \tag{4.3}$$

Using the method of Crandall[2], we derived the following fast-converging series for numerical calculation at positive integer arguments within the domain of (not necessarily absolute) convergence:

$$\zeta(s) = \frac{1}{(s-1)!} \Big\{ \sum_{n=-1}^{\infty} \beta_n \frac{1}{s+n} + \sum_{j=0}^{s-1} \sum_{n=1}^{\infty} \frac{(s-1)!}{j!} \frac{e^{-n}}{n^{s-j}} \Big\} \tag{4.4}$$

$$L_\chi(s) = \frac{1}{(s-1)!} \Big\{ \sum_{n=0}^{\infty} \beta_{n,\chi} \frac{g^{n+s}}{(n+s)} + \sum_{n=1}^{\infty} \chi(n) \sum_{j=0}^{s-1} \frac{(s-1)!}{j!} g^j \frac{e^{-gn}}{n^{s-j}} \Big\} \tag{4.5}$$

Here, and in the following formulas we choose $g \in (0, 2\pi/D)$. For our calculations, $D = 3$, so it sufficed to use $g = 1$.

We will illustrate Crandall's method in finding a series for $L(\chi, 1; a, b)$ where χ has conductor D, $a, b \in \mathbf{Z}_+$, $b > 1$. We first multiply by $\Gamma(a)\Gamma(b)$ to obtain

$$\sum_{0<m<n} \frac{\chi(m)}{m^a n^b} \int_0^\infty \int_0^\infty t^{a-1} u^{b-1} e^{-t} e^{-u} \, dt \, du$$

Since the sum is absolutely convergent (by Zhao's domain of absolute convergence), we can write this as

$$\sum_{m,n=1}^{\infty} \frac{\chi(m)}{m^a (m+n)^b} \int_0^\infty \int_0^\infty t^{a-1} u^{b-1} e^{-t} e^{-u} \, dt \, du$$

By a standard change of variables, this is

$$\sum_{m,n=1}^{\infty} \chi(m) \int_0^\infty \int_0^\infty t^{a-1} u^{b-1} e^{-mt} e^{-(m+n)u} \, dt \, du$$

$$= \sum_{m,n=1}^{\infty} \chi(m) \int_0^\infty \int_0^\infty t^{a-1} u^{b-1} e^{-m(t+u)} e^{-nu} \, dt \, du$$

We now make the change of variables $v = u$, $w = t + u$. Since the Jacobian is 1, this is

$$\sum_{m,n=1}^{\infty} \chi(m) \int_{0<v<w} (w-v)^{a-1} v^{b-1} e^{-mw} e^{-nv} \, dv \, dw \tag{4.6}$$

$$= \sum_{k=0}^{a-1} (-1)^k \binom{a-1}{k} S_k$$

where

$$S_k = \sum_{m,n=1}^{\infty} \int_0^{\infty} \int_0^{w} v^{b+k-1} w^{a-k-1} e^{-mw} e^{-nv} \, dv \, dw \tag{4.7}$$

We now split up the region of integration into three regions. Define

$$A = \{v, w | 0 < v < w < g\}$$

$$B = \{v, w | 0 < v < g < w\}$$

$$C = \{v, w | g < v < w\}$$

where g is chosen such that $0 < g < 2\pi/D$. Now $\int_A S_k$ equals

$$\sum_{m,n=1}^{\infty} \chi(m) \int_0^{g} \int_0^{w} v^{b+k-1} w^{a-k-1} e^{-mw} e^{-nv} \, dv \, dw$$

Making use of Bernoulli numbers, this is

$$\sum_{\substack{m=0 \\ n=-1}}^{\infty} \beta_{m,\chi} \beta_n \int_0^{g} \int_0^{w} v^{b+k+n-1} w^{a-k+m-1} \, dv \, dw$$

The straightforward integration then gives

$$\int_A S_k = \sum_{\substack{m=0 \\ n=-1}}^{\infty} \frac{\beta_{m,\chi} \beta_n}{(a+b+m+n)(b+k+n)} g^{a+b+m+n} \tag{4.8}$$

Next, $\int_B S_k$ equals

$$\sum_{m,n=1}^{\infty} \chi(m) \int_g^{\infty} \int_0^{g} v^{b+k-1} w^{a-k-1} e^{-mw} e^{-nv} \, dv \, dw$$

Again using Bernoulli numbers, this equals

$$\sum_{m=1}^{\infty} \chi(m) \int_1^{\infty} w^{a-k-1} e^{-mw} \, dw \cdot \sum_{n=-1}^{\infty} \beta_n \int_0^{1} v^{b+k+n-1} \, dv$$

We evaluate the simple integral on the right, and evaluate the integral on the left by making use of the integration formula

$$\int t^l e^{-ct} \, dt = -e^{-ct} \sum_{k=0}^{l} \frac{l!}{k!} \frac{t^k}{c^{l-k+1}} \tag{4.9}$$

to obtain

$$\int_B S_k = \sum_{m=1}^{\infty} \chi(m) \sum_{j=0}^{a-k-1} \frac{(a-k-1)!}{j!} \frac{g^k e^{-mg}}{m^{a-k-j}} \cdot \sum_{n=-1}^{\infty} \frac{\beta_n}{b+k+n} \tag{4.10}$$

To find $\int_C S_k$, we again use the integration formula (4.9) repeatedly:

$$\int_C S_k = \sum_{m,n=1}^{\infty} \int_g^{\infty} \int_g^{w} v^{b+k-1} w^{a-k-1} e^{-mw} e^{-nv} \, dv \, dw$$

$$
\begin{aligned}
&= \sum_{m,n=1}^{\infty} \chi(m) \int_g^{\infty} w^{a-k-1} e^{-mw} \sum_{j_1=0}^{b+k-1} \frac{(b+k-1)!}{j_1!} (-e^{-nv}) \frac{v^{j_1}}{n^{b+k-j_1}} \Big|_{v=g}^{w} \, dw \\
&= \sum_{m,n=1}^{\infty} \chi(m) \sum_{j_1=0}^{b+k-1} \frac{(b+k-1)!}{j_1!} \int_g^{\infty} w^{a-k-1} e^{-mw} \Big\{ \frac{g^{j_1} e^{-ng}}{n^{b+k-j_1}} - \frac{w^{j_1} e^{-nw}}{n^{b+k-j_1}} \Big\} \, dw \\
&= \sum_{m,n=1}^{\infty} \chi(m) \sum_{j_1=0}^{b+k-1} \frac{(b+k-1)!}{j_1!} \Big\{ \frac{g^{j_1} e^{-ng}}{n^{b+k-j_1}} \int_g^{\infty} w^{a-k-1} e^{-mw} \, dw \\
&\qquad - \frac{1}{n^{b+k-j_1}} \int_g^{\infty} w^{a-k+j_1-1} e^{-(m+n)w} \, dw \Big\} \\
&= \sum_{m,n=1}^{\infty} \chi(m) \sum_{j_1=0}^{b+k-1} \frac{(b+k-1)!}{j_1!} \frac{1}{n^{b+k-j_1}} \Big\{ g^{j_1} e^{-ng} \sum_{j_2=0}^{a-k-1} \frac{(a-k-1)!}{j_2!} \cdot \\
&\qquad \frac{g^{j_2} e^{-mg}}{m^{a-k-j_2}} - \sum_{j_3=0}^{a-k+j_1-1} \frac{(a-k+j_1-1)!}{j_3!} \frac{g^{j_3} e^{-(m+n)g}}{(m+n)^{a-k+j_1-j_3}} \Big\} \\
&= \sum_{m,n=1}^{\infty} \chi(m) e^{-(m+n)g} \sum_{j_1=0}^{b+k-1} \frac{(b+k-1)!}{j_1!} \frac{1}{n^{b+k-1}} \Big\{ \sum_{j_2=0}^{a-k-1} \frac{(a-k-1)!}{j_2!} \cdot \\
&\qquad \frac{g^{j_1+j_2}}{m^{a-k-j_2}} - \sum_{j_3=0}^{a-k+j_1-1} \frac{(a-k+j_1-1)!}{j_3!} \frac{g^{j_3}}{(m+n)^{a-k+j_1-j_3}} \Big\}
\end{aligned}
\tag{4.11}
$$

Adding up $\sum_{k=0}^{a-1} (-1)^k \binom{a-1}{k} \int S_k$ over each of the three pieces, we obtain the resulting series for $L(\chi, 1; a, b)$

$$
\begin{aligned}
& L\binom{\chi, 1}{s_1, s_2} = \frac{1}{(s_1-1)!} \frac{1}{(s_2-1)!} \sum_{j=0}^{s_1-1} (-1)^j \binom{s_1-1}{j} \cdot \\
& \Big\{ \sum_{\substack{m=0 \\ n=-1}}^{\infty} \beta_{m,\chi} \beta_n \frac{1}{s_2+n+j} \frac{g^{s_1+s_2+m+n}}{s_1+s_2+m+n} \\
& + \sum_{j_1=0}^{s_1-j-1} \frac{(s_1-j-1)!}{j_1!} \sum_{\substack{m=1 \\ n=-1}} \beta_n \chi(m) \frac{g^{s_2+n+j}}{s_2+n+j} \frac{e^{-gm}}{m^{s_1-j-j_1}} \\
& + \sum_{j_1=0}^{s_2+j-1} \frac{(s_2+j-1)!}{j_1!} \sum_{m,n=1}^{\infty} \frac{\chi(m) e^{-g(m+n)}}{n^{s_2+j-j_1}} \Big\{ \sum_{j_2=0}^{s_1-j-1} \frac{(s_1-j-1)!}{j_2!} \frac{g^{j_1+j_2}}{m^{s_1-j-j_2}} \\
& - \sum_{j_3=0}^{s_1-j+j_1-1} \frac{(s_1-j+j_1-1)!}{j_3!} \frac{g^{j_3}}{(m+n)^{s_1-j+j_1-j_3}} \Big\} \Big\}
\end{aligned}
\tag{4.12}
$$

In order to calculate numerically numbers $Li_{1,1}(\zeta^{-1}, \zeta)$, for $\zeta \neq 1$ a root of unity, a somewhat different method is necessary, due to the conditional convergence of the double Dirichlet series. By integrating geometric series, we get the integral representation

$$
\int_0^{\zeta} \int_0^{\zeta^{-1}} \frac{u}{(1-u)(1-tu)} \, dt \, du \tag{4.13}
$$

Making the change of variables $t = \zeta^{-1} t'$, $u = \zeta u'$, this is

$$
\int_0^1 \int_0^1 \frac{\zeta u}{(1-\zeta u)(1-tu)} \, dt \, du \tag{4.14}
$$

Now we make another change of variables $t' = e^{-t}$, $u' = e^{u}$. We obtain

$$\zeta \int_0^\infty \int_0^\infty \frac{e^{-t-2u}}{(1-\zeta e^{-u})(1-e^{-t-u})}\, dt\, du \tag{4.15}$$

A last change of variables $w_1 = u$, $w_2 = t + u$ gives

$$\zeta \int_{0<w_1<w_2} \frac{e^{-w_1-w_2}}{(1-\zeta e^{-w_1})(1-e^{-w_2})}\, dw \tag{4.16}$$

We can expand the integrand in geometric series and use Crandall's method to evaluate this integral also, giving the series

$$\begin{aligned} Li_{1,1}(\zeta^{-1},\zeta) = {} & \sum_{\substack{m=1\\n=0}}^{\infty} \frac{\alpha_{m-1}(\zeta^{-1})\beta_{n-1}}{m(m+n)} g^{m+n} + \sum_{n=1}^{\infty} \frac{e^{-ng}}{n} \cdot \sum_{m=1}^{\infty} \alpha_{m-1}(\zeta^{-1}) \frac{g^m}{m} \\ & + \sum_{m,n=1}^{\infty} \zeta^n \frac{e^{-(m+n)g}}{m(m+n)} \end{aligned} \tag{4.17}$$

where

$$\alpha_m(\sigma) = \sum_{k=1}^{D} \sigma^k \sum_{n=0}^{m+1} \frac{(D-k)^n}{n!} \beta_{m-n} D^{m-n} \quad \text{and} \quad 0 < g < \frac{2\pi}{D} \tag{4.18}$$

with σ a generator of μ_D.

Another example of a value whose defining sum converges only conditionally, and which lends itself to a similar series is $Li_{s,1}(1,\zeta)$, for $s > 1$ and ζ a root of unity. We find

$$\Gamma(s) Li_{s,1}(1,\zeta) = \sum_{m,n=1}^{\infty} \zeta^{m+n} \int_{0<w_1<w_2} (w_2-w_1)^{s-1} e^{-mw_2} e^{-nw_1}\, dw$$

Using methods similar to the above, we obtain the series

$$\begin{aligned} Li_{s,1}(1,\zeta) = {} & \frac{1}{(s-1)!} \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \Bigg\{ \sum_{m,n=0}^{\infty} \frac{\alpha_m(\zeta)\alpha_n(\zeta) g^{m+n}}{(s+m+n+1)(k+n+1)} \\ & + \sum_{n=0}^{\infty} \frac{\alpha_n(\zeta) g^n}{k+n+1} \sum_{m=1}^{\infty} \sum_{j=0}^{s-k-1} \frac{(s-k-1)!}{j!} \frac{g^j (\zeta e^{-g})^m}{m^{s-k-j}} + \sum_{m,n=1}^{\infty} (\zeta e^{-g})^{m+n} \\ & \sum_{j_1=0}^{k} \frac{k!}{j_1!} \frac{1}{n^{k+1-j_1}} \Bigg\{ \sum_{j_2=0}^{s-k-1} \frac{(s-k-1)!}{j_2!} \frac{g^{j_1+j_2}}{m^{s-k-j_2}} \\ & - \sum_{j_3=0}^{s-k+j_1-1} \frac{(s-k+j_1-1)!}{j_3!} \frac{g^{j_3}}{(m+n)^{s-k+j_1-j_3}} \Bigg\} \Bigg\} \end{aligned} \tag{4.19}$$

with $\alpha_m(\zeta)$ defined as before.

Using N as the upper limit for each infinite series, Crandall gives error terms $(g/2\pi)^{-N}$ for the pieces in which Bernoulli numbers are used (which come from the integrals over bounded regions), and e^{-N} for the other pieces (which come from integrals over the infinite regions). In practice, we used stabilization of the computed numbers as the criterion for accuracy; however, we noticed that Crandall's bounds for the errors seemed accurate.

We used the PARI command `lindep` to find evaluations numerically from numbers computed using Crandall's method. This command takes as input a vector $x \in \mathbf{C}^n$ and, optionally, a parameter specifying accuracy, and outputs a vector $m \in \mathbf{Z}^n$ such that $m \cdot x$ is small. If the accuracy parameter is present, the LLL algorithm due to Hastad, Helfrich, Lagarias, and Schnorr is used. If this parameter is not present, a variation of this algorithm is used. The LLL algorithm works by searching for integer vectors m such that $m \cdot x$ is small, for m with 2-norm less than or equal to some bound. If none exist, this bound is increased, and the search procedure is repeated. Thus, a drawback of the use of this command is that the output is not necessarily the 'smallest' integer relation among the numbers in the vector x; all that is promised is a relation involving integers that are 'not too large.' Hence, an evaluation produced numerically using the `lindep` command may not be exactly the same as the 'nicest' one we can derive using the methods of our proofs.

5. Appendix: A Sample Computational Session

In this section, we give an example of a numerical verification of an explicit evaluation. Here is the code for a series for numerically computing $L(\chi, 1; s_1, s_2)$ for χ a Dirichlet character and $s_2 > 1$:

```
{bernnew(n)=bernreal(n+1)/(n+1)!}

{k(D,n)=kronecker(D,n)}

{bernchar(D,n)=sum(a1=1,abs(D)-1,k(D,a1)*sum(m=0,n+1,
(abs(D)-a1)**m/m!*abs(D)**(n-m)*bernnew(n-m)))}

{g(D)=Pi/abs(D)}

{sumdbl(x,y,A,B,N,sum1,sum2)=sum1=0.;sum2=0.;
for(q1=1,N,
   sum2=sum2+x**q1/q1**A;
   sum1=sum1+y**(q1+1)/(q1+1)**B*sum2);
sum1}

{dbllseries(s1,s2,D,N1,N2,m,n,j,k1,k2,G)=
Gausssum=sum(k2=1,abs(D)-1,k(D,k2)*a(D)**k2);
v=vector(N1+2,x,bernnew(x-2));
vc=vector(N1+1,x,bernchar(D,x-1));
sum(j=0,s1-1,(-1)**j(s1-1)!/j!/(s1-j-1)!*
   (sum(m=0,N1,
      sum(n=-1,N1,vc[m+1]*v[n+2]*g(D)**(s1+s2+m+n)/(s1
      +s2+m+n)/(s2+j+n)))
   +sum(n=-1,N1,v[n+2]*g(D)**(s2+n+j)/(s2+n+j))*
      sum(j1=0,s1-j-1,(s1-j-1)!/j1!*g(D)**j1*
         sum(m=1,N2,k(D,m)*exp(-g(D)*m)/m**(s1-j-j1)))
   +sum(j1=0,s2+j-1,(s2+j-1)!/j1!*(sum(j2=0,s1-j-1,(s1-j-1)
      !/j2!*g(D)**(j1+j2)*
      sum(m=1,N2,k(D,m)*exp(-g(D)*m)/m**(s1-j-j2))*
         sum(n=1,N2,exp(-g(D)*n)/n**(s2+j-j1)))
   -sum(j3=0,s1-j+j1-1,(s1-j+j1-1)!/j3!*g(D)**j3*
      sum(k1=1,abs(D)-1,k(D,k1)*
         sumdbl(q(D)**(-k1),q(D)**k1*
            exp(-g(D)),s2+j-j1,s1-j+j1-j3,N2))/Gausssum)))))
/(s1-1)!/(s2-1)!}
```

The command `bernnew` redefines the Bernoulli numbers to our definition; the command `bernchar` does the same for the Bernoulli numbers of a character. The command `sumdbl` is a subroutine facilitating the Bailey acceleration exposited in [2]. The `dbllseries` command computes the double L-value with upper limits $N1$ and $N2$, using the method of Crandall.

In conclusion, we give a session verifying the evaluation (2.2).

```
?  default(realprecision,80)
realprecision=86 significant digits (80 digits displayed)
?  dbllseries(2,2,-3,200,400)
%1 0.554075058010439745649483726026241844251690508101901
722682057224435736724658797 - 6.70254894 E-88*I
?  z2=Pi**2/6
%2 1.644934066848226436472415166646025186218949901206798
4377355582293700074704032008
?  z3=rzeta(3,200,400)
%3 1.202056903159594285399738161511449990764986292340498
8817922715553418382057863130
?  l1=lser(1,-3,200,400)
```

```
%4 0.604599788078072616864692752547385244094688749364246
85852329497881737740721972861
?  l2=lser(2,-3,200,400)
%5 0.781302412896486296867187429624092356365134336545285
42022210006333647205086362996
?  l3=lser(3,-3,200,400)
%6 0.884023811750079856743057916871011807747946186117658
93478258741494168416628016992
?  l4=lser(4,-3,200,400)
%7 0.940025680877123768691069445070885991643800309660335
012024137217271135803965594510
?  lindep([%1,l4,z2*l2,z3*l1],50)
%18 = [-9,9,12,-26]
```

Here `rzeta` and `lser` are commands which compute values of the Riemann zeta function and L-series using the method of Crandall (for which the series were given in section 7). We used upper limits $N1 = 200$ for the Bernoulli number series and $N2 = 400$ for the exponential series. The reason for this was to counteract the lesser accuracy of the latter; we also could have adjusted the choice of g to balance the two. Observing the stabilization of the numbers, the apparent accuracy is 70 to 80 decimal places. We can see that the result of the `lindep` command in the last line verifies (2.2), which we repeat here:

$$L\binom{\chi, 1}{2, 2} = L_\chi(4) + \frac{4}{3}\zeta(2)L_\chi(2) - \frac{26}{9}L_\chi(1)\zeta(3)$$

In this formula, χ is the quadratic character of conductor 3. An emacs file containing PARI code for these commands and other similar ones can be downloaded from the website:

`http://www.math.psu.edu/terhune/`

6. Bibliography

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