Assignment 7
Analytic Number Theory
MATH 773
Spring 2006
Due 5 May

1. One can define Bernoulli polynomials associated to a Dirichlet character χ via the generating function

$$B_\chi(X,t) := te^{Xt}\sum_{n=1}^{\infty}\chi(n)e^{-nt} = \sum_{n=0}^{\infty}B_{n,\chi}(X)\frac{t^n}{n!},$$

where the first sum converges for $\Re(t) > 0$, and the power series on the RHS converges for $|t| < 2\pi/m$, where m is the conductor of χ. These were first defined by Leopoldt in 1958. We will assume $\chi \neq \chi_0$.

   (a) First write $B_\chi(X,t)$ as a finite sum by summing over residue classes, and using geometric series. Show the resulting sum has a power series valid for $|t| < 2\pi/m$.

   (b) Prove the formula

$$B_{n,\chi}(-X) = \chi(-1)(-1)^n B_{n,\chi}(X).$$

   (c) Prove the identity

$$\sum_{k=1}^{m}\chi(k)k^n = \frac{1}{n+1}\Big\{\chi(-1)B_{n+1,\chi}(m) + (-1)^n B_{n+1,\chi}(0)\Big\},$$

   where $\chi \neq \chi_0$, and $m = \text{cond }(\chi)$. *Hint: You might need to use part (b).*

Proof. By dividing up into residue classes, and summing geometric series,

$$te^{Xt}\sum_{k=1}^{m}\chi(k)\sum_{\substack{n>0\\ n\equiv k \pmod m}}e^{-nt} = te^{Xt}\sum_{k=1}^{m}\chi(k)\sum_{n=0}^{\infty}e^{-(mn+k)t}$$

$$= \frac{te^{Xt}}{1-e^{-mt}}\sum_{k=1}^{m}\chi(k)e^{-kt}.$$

Since the singularities closest to zero are at $\pm 2\pi i/m$, the power series has radius of convergence $2\pi/m$. This is (a). For (b), note

$$B_\chi(-X,-t) = \frac{-te^{Xt}}{1-e^{mt}}\sum_{k=1}^{m}\chi(k)e^{kt} = \frac{te^{Xt}}{1-e^{-mt}}\sum_{k=1}^{m}\chi(k)e^{(k-m)t}.$$

Then, by the change of index $k \mapsto m-k$, this is

$$\chi(-1)\frac{te^{Xt}}{1-e^{-mt}}\sum_{k=1}^{m}\chi(k)e^{-kt} = \chi(-1)B_\chi(X,t).$$

So by equating power series coefficients,

$$B_{n,\chi}(-X) = (-1)^n\chi(-1)B_{n,\chi}(X).$$

For (c), note

$$f_\chi(0,t) - f_\chi(-m,t) = t\sum_{k=1}^{m}\chi(k)e^{-kt}.$$

Then, by equating power series coefficients,

$$B_{n+1,\chi}(0) - B_{n+1,\chi}(-m) = \sum_{k=1}^{m}\chi(k)(-k)^n(n+1)$$

$$= (n+1)(-1)^n\sum_{k=1}^{m}\chi(k)k^n.$$

Now, we use part (b), which tells us

$$B_{n+1,\chi}(-m) = (-1)^{n+1}\chi(-1)B_{n+1,\chi}(m),$$

and gives the result. □

2. Recall the Euler-Maclauren summation formula

$$\sum_{t=m+1}^{n} f(t) = \int_m^n f(t)\,dt + \sum_{j=1}^{k}(-1)^j\frac{B_j}{j!}\left(f^{(j-1)}(n) - f^{(j-1)}(m)\right)$$

$$+\frac{(-1)^{k-1}}{k!}\int_m^n B_k(t-[t])f^{(k)}(t)\,dt,$$

where $f \in C^k[m,n]$, the $\{B_j\}$ are the Bernoulli numbers, the $\{B_j(t)\}$ are the Bernoulli polynomials, and $m,n \in \mathbb{Z}$. Apply this formula to the function $f(t) = t^a$, where $a \in \mathbb{Z}_+$, to prove the identity

$$\sum_{j=1}^{n} j^a = \frac{1}{a+1}\Big\{B_{a+1}(n+1) + (-1)^a B_{a+1}\Big\}.$$

Hint: Use theorem 12.12, and the fact $B_n(1-X) = (-1)^n B_n(X)$.

Proof. Using the Euler-Maclauren formula with $k = a+1$, we have

$$\sum_{j=1}^{n} j^a = \int_0^n t^a,dt + \sum_{j=1}^{a+1}(-1)^j\frac{B_j}{j!}a(a-1)\cdots(a-j+2)n^{a-j+1}$$

$$+(-1)^a \frac{B_{a+1}}{a+1} + 0.$$

Using binomial coefficients, this is

$$\frac{n^{a+1}}{a+1} + \sum_{j=1}^{a+1} \frac{(-1)^j B_j}{a+1} \binom{a+1}{j} n^{a-j+1} + \frac{(-1)^a}{a+1} B_{a+1}$$

$$= \frac{1}{a+1} \sum_{j=0}^{a+1} (-1)^j B_j \binom{a+1}{j} n^{a-j+1} + \frac{(-1)^a}{a+1} B_{a+1}$$

$$= \frac{(-1)^{a+1}}{a+1} \sum_{j=0}^{a+1} B_j \binom{a+1}{j} (-n)^{a-j+1} + \frac{(-1)^a}{a+1} B_{a+1}.$$

Now, by theorem 12.12, this is

$$\frac{(-1)^{a+1}}{a+1} B_{a+1}(-n) + \frac{(-1)^a}{a+1} B_{a+1},$$

which by the identity $B_n(1-X) = (-1)^n B_n(X)$, gives

$$\frac{1}{a+1} \Big(B_{a+1}(n+1) + (-1)^a B_{a+1} \Big).$$

$\square$