Solutions - Assignment 5
Analytic Number Theory
MATH 773
Spring 2006

1. (problem 4, ch. 7 in text) For $h, k \in \mathbb{Z}_+$, we define

$$A(h,k) := \{h + kx : x \in \mathbb{Z}, x \geq 0\}.$$

Let $S \subset A(h,k)$ be any infinite subset. Show that for any $n \in \mathbb{Z}_+$, some element of $A(h,k)$ can be expressed as a product of more than n distinct elements of S (the book says you are not allowed to use Dirichlet's theorem for this, but I don't know what good it would do you).

Proof. Let m be the order of h modulo k, $n \in \mathbb{Z}_+$. Then for any distinct $x_1, \ldots, x_{nm+1} \in S$,

$$\prod_{j=1}^{nm+1} x_j \in A(h,k).$$

So there exist arbitrarily large $m \in \mathbb{Z}_+$, such that for some distinct $x_1, \ldots, x_m \in S$, $\prod_{j=1}^{m} x_j \in A(h,k)$. But this clearly contradicts what it means for the problem to be false, which is 'there exists $n \in \mathbb{Z}_+$ such that no element of $A(h,k)$ is a product of $n+1$ or more distinct elements of S.' □

2. (problem 6, ch. 7 in text) Let $k \in \mathbb{Z}_+$, $(h,k) = 1$. Show that

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{1}{p} = \frac{1}{\varphi(k)} \log\log x + A + O\Big(\frac{1}{\log x}\Big),$$

for some constant A. Also give a value for A.

Proof. We use theorem 7.3, which states

$$A(x) := \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + O(1).$$

We apply Abel's identity with $f(t) = 1/\log t$, giving

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{1}{p} = \frac{A(x)}{\log x} + \int_{3/2}^{x} \frac{A(t)}{t \log^2 t}\, dt.$$

The first term is $1/\varphi(k) + O(1/\log x)$. The second is

$$\int_{3/2}^{x} \frac{A(t) - \log t/\varphi(k)}{t \log^2 t}\, dt + \frac{1}{\varphi(k)} \int_{3/2}^{x} \frac{dt}{t \log t}\, dt. \qquad (1)$$

The first integral here is

$$\int_{3/2}^{\infty} \frac{A(t) - \log t/\varphi(k)}{t \log^2 t}\, dt - \int_{x}^{\infty} \frac{A(t) - \log t/\varphi(k)}{t \log^2 t}\, dt.$$

The first term above is convergent, since the numerator is bounded. Similarly, the second term is $O(1/\log x)$. The second term of (1) is easily evaluated, and equals

$$\frac{1}{\varphi(k)}\bigl(\log\log x - \log\log\frac{3}{2}\bigr).$$

Putting all of this together, we have

$$\sum_{\substack{p\le x \\ p\equiv h \pmod k}} \frac{1}{p} = \frac{1}{\varphi(k)} \log\log x + A + O\Bigl(\frac{1}{\log x}\Bigr),$$

where

$$A = \frac{1}{\varphi(k)}\bigl(1 - \log\log\frac{3}{2}\bigr) + \int_{3/2}^{\infty} \frac{A(t) - \log t/\varphi(k)}{t \log^2 t}\, dt.$$

□

3. (problems 7, 8, ch. 11 in text) For $n = p_1^{a_1} \cdots p_k^{a_k}$ a factorization into distinct primes, set $\nu(n) = k$. Show the following identities hold for $\sigma > 1$:

$$\sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}, \qquad \sum_{n=1}^{\infty} \frac{2^{\nu(n)}\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)^2}.$$

Proof. Note for $n \in \mathbb{Z}_+$, $n \ge 2^{\nu(n)}$, so the first series is dominated by $\sum 1/n^{-(s-1)}$. Hence, the first series converges absolutely for $\sigma > 2$. The first identity we wish to show is equivalent to

$$\zeta(2s) \sum \frac{2^{\nu(n)}}{n^s} = \zeta(s)^2.$$

By theorem 11.5, we can accomplish this for $\sigma > 2$ by showing for each $n \in \mathbb{Z}_+$, $(f * g)(n) = d(n) = \sigma_0(n)$, where

$$f(n) = \begin{cases} 1, & n \text{ is a square} \\ 0, & \text{otherwise} \end{cases},$$

and $g(n) = 2^{\nu(n)}$. If $(m,n) = 1$, it is clear that $\nu(mn) = \nu(m) + \nu(n)$. Then, by exponentiating, g is multiplicative. Now, since f, g are both multiplicative, it is enough to verify the identity at prime powers

$$(f * g)(p^k) = \sum_{j=0}^{k} f(p^j) g(p^{k-j}) = \sum_{\substack{j=0 \\ 2|j}}^{k} g(p^{k-j}) = 1 + \sum_{\substack{j=0 \\ 2|j}}^{k-1} g(p^{k-j}).$$

By considering separately the cases k odd and even, this is seen to equal $k+1 = d(p^k)$. To show the first identity actually holds for $\sigma > 1$, we use theorem 11.13, which says that a Dirichlet series with positive coefficients has its abcissa of convergence at its rightmost real singularity, which by what we have just shown, is at $s = 1$.

Since the series on the LHS of the first identity is absolutely convergent for $\sigma > 1$, and dominates the series on the LHS of the second identity (because $|\lambda(n)| = 1$), that series also converges absolutely for $\sigma > 1$. Again, since $d, g\lambda$ are multiplicative, it suffices to show $d * g\lambda = f$ at prime powers. We compute

$$(d * g\lambda)(p^k) = \sum_{j=0}^{k} d(p^j) g(p^{k-j}) \lambda(p^{k-j})$$

$$= \sum_{j=0}^{k} (j+1) g(p^{k-j}) \lambda(p^{k-j})$$

$$= k + 1 + \sum_{j=0}^{k-1} (j+1) g(p^{k-j}) \lambda(p^{k-j})$$

$$= k + 1 + 2 \sum_{j=0}^{k-1} (j+1)(-1)^{k-j}$$

$$= k + 1 + 2(-1)^k \sum_{j=0}^{k-1} (-1)^j (j+1)$$

$$= k + 1 + 2(-1)^k (-1)^{k-1} \left[\frac{k+1}{2}\right] = k + 1 - 2\left[\frac{k+1}{2}\right].$$

Again, considering the cases k even and odd shows this

$$= \begin{cases} 1, & 2 | k \\ 0, & 2 \nmid k \end{cases} = f(p^k).$$

□