Solutions 4
Analytic Number Theory
MATH 773
Spring 2006

1. (ch. 4, problem 25 in text) Show that the prime number theorem in the form $\psi(x) \sim x$ implies Selberg's asymptotic formula of theorem 4.18, but with error term replaced by $o(x \log x)$. I.e., show PNT implies

$$\lim_{x\to\infty} \left\{ \frac{\psi(x)}{x} + \frac{1}{x \log x} \sum_{n \leq x} \Lambda(n) \psi\left(\frac{x}{n}\right) \right\} = 2.$$

Proof. It is enough to show

$$\sum_{n \leq x} \Lambda(n) \psi\left(\frac{x}{n}\right) = x \log x + o(x \log x),$$

or, i.e.,

$$\lim_{x\to\infty} \frac{1}{x \log x} \sum_{n \leq x} \Lambda(n) \psi\left(\frac{x}{n}\right) = 1.$$

Let $\varepsilon > 0$. Choose $c > 0$ such that

$$1 - \varepsilon < \frac{\psi(x)}{x} < 1 + \varepsilon \quad \text{whenever} \quad x \geq c.$$

We write

$$\sum_{n \leq x} \Lambda(n) \psi\left(\frac{x}{n}\right) = \sum_{n \leq x/c} + \sum_{x/c < n \leq x} \tag{1}$$

Regarding the first sum, we have

$$x(1-\varepsilon) \sum_{n \leq x/c} \frac{\Lambda(n)}{n} \leq x \sum_{n \leq x/c} \frac{\Lambda(n)}{n} \frac{\psi(x/n)}{x/n} \leq x(1+\varepsilon) \sum_{n \leq x/c} \frac{\Lambda(n)}{n},$$

since $n \leq x/c$ means exactly $x/n \geq c$. Now, by theorem 4.9,

$$\sum_{n \leq x/c} \frac{\Lambda(n)}{n} = \log \frac{x}{c} + O(1) = \log x + O(1),$$

for $x \geq c$. So dividing by $x \log x$ and letting $x \to \infty$, we have

$$1 - \varepsilon \leq \liminf_{x\to\infty} \frac{1}{x \log x} \sum_{n \leq x/c} \Lambda(n) \psi(x/n)$$
$$\leq \limsup_{x\to\infty} \frac{1}{x \log x} \sum_{n \leq x/c} \Lambda(n) \psi(x/n) \leq 1 + \varepsilon.$$

We now consider the second sum in (1).

$$\sum_{x/c<n\le x} \Lambda(n)\psi(x/n) \le \psi(c) \sum_{x/c<n\le x} \Lambda(n)$$
$$= \psi(c)\big(\psi(x) - \psi(x/c)\big),$$

which is $O(x)$, by the PNT. Hence, for some $A > 0$, $x > A$ implies

$$\frac{1}{x\log x} \sum_{x/c<n\le x} \Lambda(n)\psi(x/n) < \varepsilon.$$

Thus,

$$1-\varepsilon \le \liminf_{x\to\infty} \frac{1}{x\log x}\sum_{n\le x}\Lambda(n)\psi(x/n)$$
$$\le \limsup_{x\to\infty} \frac{1}{x\log x}\sum_{n\le x}\Lambda(n)\psi(x/n) \le 1+2\varepsilon.$$

Since this holds for every $\varepsilon > 0$, the two limits both equal 1, as $x \to \infty$. □

2. (ch. 6, part of problem 15 in text) Let χ be an odd character of conductor k. Show that for any integers $a < b$,

$$|\sum_{n=a}^{b} \chi(n)| \le \frac{1}{2}\varphi(k)$$

Proof. Recall $2|\varphi(k)$ for $k \ge 3$ (which is a valid assumption, since there are no odd characters of conductors $1, 2$). We observe that for these k, half of the integers j such that $1 \le j \le k$ and $(j,k) = 1$ lie in the interval $[1, k/2]$, and the other half lie in $(k/2, k]$. This is true because $(j,k) = 1$ if and only if $(j, k-j) = 1$, which follows from the definition of g.c.d.

Since $\sum_{j=1}^{k} \chi(j) = 0$, by periodicity we may assume $b - a < k$. If $b - a \le k/2$, then by our observation, the problem follows. So we have reduced to the case $k/2 < b - a < k$. Note that since χ is odd,

$$\chi(j) + \chi(k-j) = \chi(j) + \chi(-j) = \chi(j) - \chi(j) = 0,$$

for any $j \in \mathbb{Z}$. Hence,

$$|\sum_{j=a}^{b} \chi(j)| = |\sum_{\substack{a\le j\le k/2 \\ k-j>b}} \chi(j)| \le \sum_{\substack{a\le j\le k/2 \\ k-j>b}} |\chi(j)| \le \frac{1}{2}\varphi(k).$$

□