Solutions 3
Analytic Number Theory
MATH 773
Spring 2006

1. (ch. 4, problem 6 in text) Show that for no polynomials $A(x)$, $B(x)$ does it hold that
$$\pi(x) = \frac{A(x)}{B(x)},$$
for all $x \in \mathbb{Z}_+$. *Hint: Use theorem 4.6 (remember you are not allowed to use the Prime Number Theorem).*

   Proof. Since there are infinitely many primes, $\pi(x) \to \infty$ as $x \to \infty$. Hence, if $\pi(x)$ equals such a rational function, then $\deg(A) > \deg(B)$. So for some $k \in \mathbb{Z}_+$, we can write $\pi(x) = x^k g(x)$, for all $x \in \mathbb{Z}_+$, where $g(x)$ approaches a positive limit as $x \to \infty$. If $[x] = n$, then by the right inequality in theorem 4.6,
$$\pi(x) = \pi(n) \leq \frac{6n}{\log n} \leq \frac{6x}{\log x},$$
for $x > e$, using the fact that $x/\log x$ is an increasing function, for $x > e$. Hence,
$$g(x) \leq \frac{1}{x^{k-1}\log x},$$
for $x \in \mathbb{Z}_+$ sufficiently large. But the RHS approaches zero, which is a contradiction. □

2. (ch. 4, problem 18 in text) Prove that the following two relations are equivalent:

   (a)
$$\pi(x) = \frac{x}{\log x} + O\Big(\frac{x}{\log^2 x}\Big)$$

   (b)
$$\theta(x) = x + O\Big(\frac{x}{\log x}\Big)$$

   Proof. First, assume a). Then by theorem 4.3,
$$\begin{aligned}\theta(x) &= x + O\Big(\frac{x}{\log x}\Big) - \int_2^x \frac{\pi(t)}{t}\,dt \\ &= x + O\Big(\frac{x}{\log x}\Big) - \int_2^x \frac{dt}{\log t} - \int_2^x \Big(\pi(t) - \frac{t}{\log t}\Big)\frac{dt}{t} \\ &\leq x + O\Big(\frac{x}{\log x}\Big) + \int_2^x \frac{dt}{\log t} + C\int_2^x \frac{dt}{\log^2 t},\end{aligned}$$

for some $C > 0$. Now,

$$\int_2^x \frac{dt}{\log t} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x \leq \frac{\sqrt{x}}{\log 2} + \frac{x - \sqrt{x}}{\log \sqrt{x}},$$

which is obviously $O(x/\log x)$. Similarly,

$$\int_2^x \frac{dt}{\log^2 t} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x \leq \frac{\sqrt{x}}{\log^2 2} + \frac{x - \sqrt{x}}{\log^2 \sqrt{x}},$$

which is also $O(x/\log x)$. Now assume b). Again by theorem 4.3,

$$\begin{aligned}\pi(x) &= \frac{x}{\log x} + O\Big(\frac{x}{\log^2 x}\Big) + \int_2^x \frac{dt}{\log^2 t} + \int_2^x \frac{\theta(t) - t}{t \log^2 t}\, dt \\ &\leq \frac{x}{\log x} + O\Big(\frac{x}{\log^2 x}\Big) + \int_2^x \frac{dt}{\log^2 t} + C \int_2^x \frac{dt}{\log^3 t},\end{aligned}$$

for some $C > 0$. The last two integrals can be bounded in the same manner as before, and each bound thus obtained is $O(x/\log^2 x)$. ☐

3. (ch. 4, problem 20 in text) Let $f : \mathbb{Z}_+ \to \mathbb{C}$ be such that

$$\sum_{p \leq x} f(p) \log p = (ax + b) \log x + cx + O(1), \quad \text{for} \quad x \geq 2.$$

Show there is some constant $A = A(f)$ such that

$$\sum_{p \leq x} f(p) = ax + (a + c)\Big(\frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t}\Big) + b \log\log x + A + O\Big(\frac{1}{\log x}\Big).$$

Hint: Use Abel's Identity, together with the fact that

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t},$$

which results from an integration by parts. The LHS of the above relation is usually called $Li(x)$, the logarithmic integral.

Proof. Let

$$A(x) = \sum_{p \leq x} f(p) \log p,$$

and

$$g(n) = \begin{cases} f(n) \log n, & n \quad \text{prime} \\ 0, & \text{otherwise} \end{cases}.$$

Hence,

$$\sum_{p \leq x} f(p) \log p = \sum_{n \leq x} g(n).$$

Now applying Abel's identity with $f(t) = 1/\log t$,

$$\sum_{p \le x} f(p) = \sum_{n \le x} \frac{g(n)}{\log n} = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t \log^2 t}\, dt$$

$$= ax + b + \frac{cx}{\log x} + O\Big(\frac{1}{\log x}\Big) + \int_2^x \frac{A(t)}{t \log^2 t}\, dt.$$

The last integral is

$$\int_2^x \frac{a\, dt}{\log t} + \int_2^x \frac{b\, dt}{t \log t} + \int_2^x \frac{c\, dt}{\log^2 t} + \int_2^x \frac{g(t)}{t \log^2 t}\, dt,$$

where $g(t) = A(t) - (at + b)\log t - ct$, so by hypothesis, $|g(t)| \le C$, for some $C > 0$. The second integral is $b(\log\log x - \log\log 2)$. The last integral involving $g(t)$ is

$$\int_2^\infty \frac{g(t)}{t \log^2 t}\, dt - \int_x^\infty \frac{g(t)}{t \log^2 t}\, dt.$$

The first integral here is convergent, since the integrand is majorized by $C/t\log^2 t$. Similarly, the last integral is $O(1/\log x)$ by majorizing and taking an antiderivative. Putting all of this information together, and using the hint, gives the result. $\square$