Solutions 2
Analytic Number Theory
MATH 773
Spring 2006

1. (ch. 2, problem 18 in text) Recall that a perfect number n is one such that the sum of the proper divisors of n is n, i.e., $\sigma_1(n) = 2n$. Show that if $2^a - 1$ is a prime, then $2^{a-1}(2^a - 1)$ is perfect.

   Proof. If $n = 2^{a-1}(2^a - 1)$, then the only primes dividing n are 2 and $2^a - 1$. Since σ_1 is multiplicative,

   $$\sigma_1(n) = \sigma_1(2^{a-1})\sigma_1(2^a - 1) = (1 + 2 + 2^2 + \cdots + 2^{a-1})(1 + 2^a - 1),$$

   since $2^a - 1$ is prime. By summing a geometric series, this equals

   $$(2^a - 1)2^a = 2n.$$

   □

2. (ch. 2, problem 14 in text) Let

   $$f : [0, 1] \cap \mathbb{Q} \to \mathbb{C}$$

   be an arbitrary function. Set

   $$F(n) = \sum_{k=1}^{n} f\Big(\frac{k}{n}\Big), \quad F^*(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f\Big(\frac{k}{n}\Big).$$

   (a) Show the Dirichlet product $\mu * F = F^*$.

   (b) Use part (a) to show

   $$\mu(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} e^{2\pi ik/n}.$$

   Proof. For $n \in \mathbb{Z}_+$, by definition,

   $$(\mu * F)(n) = \sum_{d|n} \mu(d) \sum_{k=1}^{n/d} f(dk/n). \qquad (1)$$

   We consider first only the terms with $dk/n = c$, a constant. Hence $c = r/n$, for some $r \in \mathbb{Z}_+$, with $r \leq n$. So for each d which appears, $d|r$. Since $d|n$ and $d|r$, $d|(n, r)$, and for each such d, exactly one k satisfies $dk = r$. So the above sum, restricted to d, k such that $dk = r$ is

   $$f(c) \sum_{d|(n,r)} \mu(d) = \begin{cases} f(c), & (n, r) = 1 \\ 0, & \text{else} \end{cases}.$$

Now, summing over all such c which appear, we have

$$(\mu * F)(n) = \sum_{\substack{r=1 \\ (r,n)=1}} f\Big(\frac{r}{n}\Big) = F^*(n).$$

This shows a). To show b), we put $f(x) = e^{2\pi i x}$. Then the term in (1) with $d = n$ gives simply $\mu(n)$. The terms with $d \neq n$ give

$$\mu(d) \sum_{k=1}^{n/d} \zeta^k,$$

with $\zeta \neq 1$ a primitive n/d-th root of unity. It is not hard to see all such sums are zero. $\square$

3. (ch. 3, problem 2 in text) Prove that for $x \geq 2$,

$$\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + O(1),$$

where γ denotes Euler's constant.

Proof. This problem requires problem 1a (of ch. 3), which states

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + A + O\Big(\frac{\log x}{x}\Big), \tag{2}$$

for some constant A. To show this, we apply Euler's summation formula to get

$$\int_1^x \frac{\log t}{t}\, dt + \int_1^x (t - [t]) \frac{1 - \log t}{t^2}\, dt - \frac{\log x}{x}(x - [x])$$

$$= \frac{1}{2} \log^2 x + O\Big(\frac{\log x}{x}\Big) + \int_1^\infty (t - [t]) \frac{1 - \log t}{t^2}\, dt - \int_x^\infty (t - [t]) \frac{1 - \log t}{t^2}\, dt.$$

Note the integral over $[1, \infty)$ is convergent, as it is dominated by $\int_1^\infty t^{\varepsilon - 2}\, dt$, for any $\varepsilon > 0$. Similarly, the integral over $[x, \infty)$ is $< \log x / x$ in absolute value. This shows (2). Now, for this problem, we proceed as in the weaker result in the proof of theorem 3.3:

$$\sum_{n \leq x} \frac{d(n)}{n} = \sum_{dq \leq x} \frac{1}{dq} = \sum_{d \leq x} \frac{1}{d} \sum_{d \leq x/d} \frac{1}{q} = \sum_{d \leq x} \Big(\log \frac{x}{d} + \gamma + O\big(\frac{d}{x}\big) \Big)$$

$$= \log x \sum_{d \leq x} \frac{1}{d} - \sum_{d \leq x} \frac{\log d}{d} + \gamma \sum_{d \leq x} \frac{1}{d} + O\Big(\frac{1}{x} \sum_{d \leq x} 1 \Big).$$

Applying parts 1 and 4 of theorem 3.2, together with (2), we get

$$\log x\big(\log x + \gamma + O(1/x)\big) - \Big(\frac{1}{2}\log^2 x + A + O\big(\frac{\log x}{x}\big)\Big) \\ + \gamma\big(\log x + \gamma + O(1/x)\big) + O(1),$$

which simplifies to

$$\frac{1}{2}\log^2 x + 2\gamma \log x + O(1),$$

as required. □