**Solutions 1**
**Analytic Number Theory**
**MATH 773**
**Spring 2006**

1. (ch. 2, problem 4 in text) Show that for all $n \in \mathbb{Z}_+$ with no more than 8 prime factors,
$$\varphi(n) > \frac{n}{6}.$$

*Proof.* Let $p_1, \ldots, p_k$ be the primes dividing $n$, $k \leq 8$. Then
$$\frac{\varphi(n)}{n} = \prod_{j=1}^{k}\Big(1 - \frac{1}{p}\Big), \quad k \leq 8.$$
This is
$$\geq \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} > 0.17 > \frac{1}{6}.$$
□

2. (ch. 2, problem 3 in text) Show for all $n \in \mathbb{Z}_+$,
$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}.$$

*Proof.* It is clear the LHS defines a multiplicative function of $n$. For the RHS, recall $\mu$ and $\varphi$ are both multiplicative. Since a product of multiplicative functions is again multiplicative,
$$g(n) = \mu(n)^2\varphi(n)$$
is multiplicative. Now, the RHS is $g * u$, and so by theorem 2.14 is multiplicative.

Hence, it is enough to show the statement holds for $n = p^k$, for prime $p$. The LHS is
$$\frac{p^k}{p^{k-1}(p-1)} = \frac{p}{p-1},$$
while the RHS is
$$\frac{\mu(1)^2}{\varphi(1)} + \frac{\mu(p)^2}{\varphi(p)} = 1 + \frac{1}{p-1}.$$
□

3. (ch. 2, problem 7 in text) Let $\mu(p, d)$ denote $\mu$ evaluated at the g.c.d. of $p$ and $d$. Show that
$$\sum_{d|n} \mu(d)\mu(p,d) = \begin{cases} 1, & n = 1 \\ 2, & n = p^k, k \geq 1 \\ 0, & \text{otherwise} \end{cases}. \quad (1)$$

*Proof.* Here we use a technique similar to the previous proof. The RHS defines a multiplicative function of $n$, by inspection. Note

$$\mu(p,n) = \begin{cases} p, & p|n \\ 1, & \text{otherwise} \end{cases},$$

and so is also multiplicative by inspection. Then $\mu(n)\mu(p,n)$ is multiplicative, and again by theorem 2.14, so is the LHS of (1). So it is enough to verify the statement for $n = p^k$, and $n = q^k$ for prime $q \neq p$, which is easy. □