Assignment 6
Analytic Number Theory
MATH 773
Spring 2006
Due 12 Apr

1. Let $f : \mathbb{Z}_+ \to \mathbb{C}$ be a completely multiplicative function such that the Dirichlet series
$$L_f(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$
is absolutely convergent for $\sigma > a$. For $\alpha \in \mathbb{R}$, define
$$f_\alpha(n) = \sum_{d|n} d^\alpha f(d).$$
First, show the identity
$$\sum_{n=1}^{\infty} \frac{f_\alpha(n)}{n^s} = L_f(s-\alpha)\zeta(s)$$
for $\sigma > a + \alpha$. Next, show the identity
$$\sum_{n=1}^{\infty} \frac{f(n)f_\alpha(n)}{n^s} = L_f(s)L_{f^2}(s-\alpha),$$
where $L_{f^2}(s) = \sum f(n)^2 n^{-s}$, and give a domain of absolute convergence on which this identity holds.

2. Let
$$F(s) = \frac{1}{2\pi i} \int_\gamma z^{s-1} e^{-z}\, dz,$$
where γ is the following contour:

Show that, in fact, F is entire, and
$$F(s) = \frac{e^{-i\pi s}}{\Gamma(1-s)}.$$