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Coupled singular perturbations for phase transitions

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Abstract. The $\Gamma(L^1(\Omega;\mathbb{R}^d))$-limit of the sequence

$$J_\varepsilon(u) := \frac{1}{\varepsilon}E_\varepsilon(u),$$

where E_ε is the family of anisotropic singular perturbations

$$E_\varepsilon(u) := \int_\Omega f(x,u(x),\varepsilon\nabla u(x))\,\mathrm{d}x$$

of a non-convex functional of vector-valued functions

$$E(u) := \int_\Omega f(x,u(x),\nabla u(x))\,\mathrm{d}x$$

is obtained where f is a non-negative energy density satisfying $f(x,u,0)=0$ if and only if $u \in \{a,b\}$.

Keywords: Γ-convergence, phase transitions, singular perturbations, double-well potential

1. Introduction

Let Ω be an open, bounded, Lipschitz domain of $\mathbb{R}^N$, and let W be a bulk energy density with $\{W=0\}=\{a,b\}$, where $N \in \mathbb{N}$ and a, b are real numbers, $a<b$. Starting with the work of Modica [24] within the van der Waals–Cahn–Hilliard theory of fluid-fluid phase transitions (see also [25]), singularly perturbed non-convex functionals of the form

$$I_\varepsilon(u) := \int_\Omega \left(\frac{1}{\varepsilon}W(u)+\varepsilon|\nabla u|^2\right)\mathrm{d}x, \tag{1.1}$$

have been extensively studied in the literature with the aim of obtaining selection criteria and resolve the non-uniqueness of solutions to the classical "optimal design" problem proposed by Gurtin [22]:

$$\text{Minimize} \quad \int_\Omega W(u)\,\mathrm{d}x, \quad \text{subject to a density constraint} \quad \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x = \theta a+(1-\theta)b,$$

for some $\theta \in (0,1)$. Using De Giorgi's notion of Γ-convergence [13,14] (for a comprehensive introduction to the subject, see [11]), it was shown in [24] (see also [1] and [8]) that $\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon(u) = I(u)$

with

$$I(u) := \begin{cases} K_0 \mathrm{Per}_\Omega(E) & \text{if } u = \chi_E a + (1-\chi_E) b,\ \mathcal{L}^N(E) = \theta \mathcal{L}^N(\Omega),\ u \in BV\big(\Omega; \{a,b\}\big), \\ +\infty & \text{otherwise}, \end{cases} \tag{1.2}$$

and where $K_0 := 2\int_a^b \sqrt{W(s)}\,\mathrm{d}s$, thus showing that distributions of the two phases with minimal interfacial area and given volume fraction θ are selected in the limit as $\varepsilon \to 0^+$.

For generalizations of (1.1)–(1.2) we refer to Bouchitté [7], and Owen and Sternberg [26] for the coupled problem, where the integrand of I_ε has the form $\varepsilon^{-1} f(x, u(x), \varepsilon \nabla u(x))$, and to the work of Kohn and Sternberg [23] where the study of local minimizers for (1.1) was undertaken in the case where f is convex in the last variable.

The vector-valued uncoupled problem, where $u : \Omega \to \mathbb{R}^d, \Omega \subset \mathbb{R}^N$ ($d \in \mathbb{N},\ d > 1$), was studied in [6] and [19], and here K_0 becomes

$$\widetilde{K} := \inf\left\{ \int_{-L}^{L} W\big(g(s)\big) + \left|g'(s)\right|^2 \mathrm{d}s\colon\ L > 0, g \text{ continuous and piecewise } C^1, \right.$$
$$\left. g(-L) = a,\ g(L) = b \right\}. \tag{1.3}$$

The case where W has more than two wells was addressed by Baldo [5] (see also Sternberg [28]), and later generalized by Ambrosio [2].

We adopt to the vector-valued case a framework similar to that considered by Bouchitté and by Owen and Sternberg in the scalar case. Here $I_\varepsilon(u)$ is replaced by

$$E_\varepsilon(u) := \frac{1}{\varepsilon} \int_\Omega f\big(x, u(x), \varepsilon \nabla u(x)\big)\,\mathrm{d}x,$$

and $f : \Omega \times \mathbb{R}^d \times M^{d\times N} \to [0, +\infty)$ is a continuous function satisfying the following hypotheses:

(H1) $f(x, u, 0) = 0$ if and only if $u \in \{a, b\}$;

(H2) there exists a continuous function $g : \overline{\Omega} \times \mathbb{R}^d \to [0, +\infty)$ such that

$$\frac{1}{C}\big(g(x,u) + |\xi|^2\big) \leqslant f(x,u,\xi) \leqslant C\big(g(x,u) + |\xi|^2\big)$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times M^{d\times N}$, and

$$\frac{1}{C}|u|^q - C \leqslant g(x,u) \leqslant C\big(1 + |u|^q\big)$$

for some $q \geqslant 2, C > 0$, and for all $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times M^{d\times N}$.

(H3) For any $x_0 \in \Omega$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies that

$$\big|f(x,u,\xi) - f(x_0,u,\xi)\big| \leqslant \varepsilon f(x,u,\xi)$$

for every $(u, \xi) \in \mathbb{R}^d \times M^{d\times N}$.

Before stating the Γ-convergence result of this paper, we need to introduce some notation.

Given $\nu \in S^{N-1} := \{x \in \mathbb{R}^N\colon\ \|x\| = 1\}$, we denote by Q_ν an open unit cube centered at the origin with two of its faces normal to ν, i.e., if $\{\nu_1, \dots, \nu_{N-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^N$, then

$$Q_\nu := \left\{x \in \mathbb{R}^{\mathbb{N}}\colon\ |x \cdot \nu_i| < \frac{1}{2},\ |x \cdot \nu| < \frac{1}{2},\ i = 1, \dots, N-1\right\}.$$

Also, if $x_0 \in \mathbb{R}^N$ and $r > 0$ then $Q(x_0, r) := x_0 + (-\frac{1}{2}, \frac{1}{2})^N$, i.e., $Q(x_0, r) = x_0 + rQ_{e_N}$, where $\{e_1, \dots, e_N\}$ is the standard orthonormal basis of $\mathbb{R}^N$.

Let $(a, b, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$, and define the class of admissible functions

$$\mathcal{A}(a, b, \nu) := \left\{\xi \in W^{1,\infty}_{\mathrm{loc}}(S_\nu; \mathbb{R}^d)\colon\ \xi(y) = a \text{ if } y \cdot \nu = -\frac{1}{2},\ \xi(y) = b \ \text{ if } y \cdot \nu = \frac{1}{2},\right.$$
$$\left.\text{and } \xi(y) = \xi(y + k\nu_i),\ \text{for all } y \in S_\nu,\ i \in \{1, \dots, N-1\},\ \text{and } k \in \mathbb{Z}\right\},$$

where S_ν is the strip

$$S_\nu := \left\{y \in \mathbb{R}^N\colon\ |y \cdot \nu| < \frac{1}{2}\right\},$$

and where the boundary values of ξ are understood in the sense of traces.

We introduce the surface energy density $K : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \to [0, +\infty)$ defined by

$$K(x, a, b, \nu) := \inf_{s>0} \left\{ \int_{Q_\nu} \frac{1}{s} f\bigl(x, \xi(y), s\nabla\xi(y)\bigr)\,\mathrm{d}y\colon\ \xi \in \mathcal{A}(a, b, \nu)\right\}.$$

The main result of the paper is the following theorem.

Theorem 1.1. *Assume that* (H1)–(H3) *hold, and for every* $\varepsilon > 0$ *let* $J_\varepsilon : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ *be the functional defined by*

$$J_\varepsilon(u) := \begin{cases} E_\varepsilon(u) & \text{if } u \in H^1(\Omega; \mathbb{R}^d), \\ +\infty & \text{otherwise}. \end{cases}$$

Then J_ε $\Gamma(L^1(\Omega))$*-converges to the functional* $J_0 : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ *defined by*

$$J_0(u) := \begin{cases} \displaystyle\int_{\Omega \cap \partial^* A_0} K\bigl(x, a, b, \nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x) & \text{if } u \in BV\bigl(\Omega; \{a, b\}\bigr), \\ +\infty & \text{otherwise}, \end{cases}$$

where $A_0 := \{x \in \Omega\colon\ u(x) = a\}$, *and* $\nu(x)$ *stands for the measure theoretic inner unit normal to the reduced boundary* $\partial^* A_0$ *at* x.

Remark 1.2.

(i) The Γ-convergence results obtained in the scalar case by Bouchitté [7] and by Owen and Sternberg [26] assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ is convex in the last variable, and, in addition, require some regularity of f. Precisely, in [26] f is assumed to be of class $C^3(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$, thus extending the result of Owen, who treated the case of radial perturbations (f is of class C^4 and depends radially on ξ), while [7] requires differentiability of $f(x, u, \cdot)$ at the origin where it is assumed that f achieves a strict minimum. The proof in [7] is based on duality techniques, and the expression of the Γ-limit involves the *conical envelope* of $f(x, u, \cdot)$, defined by $f_c(x, u, \cdot) := \inf_{t>0} \frac{1}{t} f(x, u, t, \cdot)$. Here, we do not assume any convexity condition on f, and although we need to impose some technical growth hypothesis (H2) (and here we recall that convex problems usually do not require growth conditions, while non-convex, vector-valued problems use them in an essential way), our regularity assumption (H3) is somewhat milder. For the proof of Theorem 1.1 we need to use different methods due to the technical nature of the vector-valued case, and to the fact that the lack of convexity does not allow us to invoke duality arguments.

(ii) Taking $f(x, u, \xi) := W(u) + h^2(x, \xi)$, we recover the result of Barroso and Fonseca (see [6]) concerning the $\Gamma(L^1)$-limit of

$$u \mapsto \int_\Omega \left(\frac{1}{\varepsilon} W\big(u(x)\big)\, \mathrm{d}x + \varepsilon h^2\big(x, \nabla u(x)\big) \right) \mathrm{d}x,$$

where W has two isolated (global) minimum points at $a, b \in \mathbb{R}^d$. Although a priori (H5) of [6] is weaker than (H3) of this paper, Lemma 2.8 and hypothesis (H5) in [6] imply that $W + (h^\infty)^2(\cdot,\cdot)$ satisfies our hypothesis (H3), where (h^∞) is the *recession function* defined by $h^\infty(x, \xi) := \limsup_{t\to\infty} \frac{h(x,t\xi)}{t}$. It can be shown that under (H4) of [6] we have

$$\begin{aligned}
&\inf\left\{ \liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_\Omega \big(W\big(u_\varepsilon(x)\big) + \varepsilon^2 h^2\big(x, \nabla u_\varepsilon(x)\big)\big)\, \mathrm{d}x : u_\varepsilon \to u \text{ in } L^1\big(\Omega; \mathbb{R}^d\big),\right.\\
&\qquad \left. u_\varepsilon \in H^1\big(\Omega; \mathbb{R}^d\big)\right\}\\
&\quad = \inf\left\{ \liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_\Omega \big(W\big(u_\varepsilon(x)\big) + \big(h^\infty\big)^2\big(x, \varepsilon\nabla u_\varepsilon(x)\big)\big)\, \mathrm{d}x : u_\varepsilon \to u \text{ in } L^1\big(\Omega; \mathbb{R}^d\big),\right.\\
&\qquad\quad \left. u_\varepsilon \in H^1\big(\Omega; \mathbb{R}^d\big)\right\}
\end{aligned}$$

and now, in view of the above considerations, we are in position to apply the result of this paper. We remark that the technique used here to prove Theorem 1.1 is different from that in [6], in particular in what concerns the existence of a recovering subsequence for the Γ-limit, where our approach in this paper is based on localization methods for Γ-convergence, De Giorgi's slicing method, and a blow-up argument which was only incipient at the time when [6] was written.

2. Preliminaries

We begin this section by briefly recalling some facts about functions of bounded variations (we refer the reader to [4] for a detailed study). A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if for all $i = 1,\dots,d$, and $j = 1,\dots,N$, there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x)\frac{\partial v}{\partial x_j}(x)\,\mathrm{d}x = -\int_\Omega v(x)\,\mathrm{d}\mu_{ij}$$

for every $v \in C_c^1(\Omega;\mathbb{R})$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}. Given $u \in BV(\Omega;\mathbb{R}^d)$ the *approximate upper* and *lower limit* of each component u_i, $i = 1,\dots,d$, are given by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R}\colon \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\mathcal{L}^N\big(\{y \in \Omega\cap Q(x,\varepsilon)\colon\ u_i(y) > t\}\big) = 0\right\}$$

and

$$u_i^-(x) := \sup\left\{t \in \mathbb{R}\colon \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\mathcal{L}^N\big(\{y \in \Omega\cap Q(x,\varepsilon)\colon\ u_i(y) < t\}\big) = 0\right\},$$

while the *jump set* of u, or *singular set*, is defined by

$$S(u) := \bigcup_{i=1}^d \{x \in \Omega\colon\ u_i^-(x) < u_i^+(x)\}.$$

It is well known that $S(u)$ is $N-1$ rectifiable, i.e.,

$$S(u) = \bigcup_{n=1}^\infty K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface. If $x \in \Omega\setminus S(u)$ then $u(x)$ is taken to be the common value of $(u_1^+(x),\dots,u_d^+(x))$ and $(u_1^-(x),\dots,u_d^-(x))$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega\setminus S(u)$. Furthermore, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$ there exists a unit vector $\nu_u(x) \in S^{N-1}$, normal to $S(u)$ at x, and two vectors $u^-(x), u^+(x) \in \mathbb{R}^d$ (the traces of u on $S(u)$ at the point x) such that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{\{y\in Q(x_0,\varepsilon):(y-x)\cdot\nu_u(x)>0\}} \left|u(y) - u^+(x)\right|^{N/(N-1)}\mathrm{d}y = 0$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{\{y\in Q(x_0,\varepsilon):(y-x)\cdot\nu_u(x)<0\}} \left|u(y) - u^-(x)\right|^{N/(N-1)}\mathrm{d}y = 0.$$

Note that, in general, $(u_i)^+ \neq (u^+)_i$ and $(u_i)^- \neq (u^-)_i$. We denote the *jump of u across* $S(u)$ by $[u] := u^+ - u^-$. The distributional derivative Du may be decomposed as

$$Du = \nabla u\mathcal{L}^N + \left(u^+ - u^-\right)\otimes\nu_u\mathcal{H}^{N-1}\lfloor S(u) + C(u),$$

where ∇u is the density of the absolutely continuous part of Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$ and $C(u)$ is the Cantor part of Du. These three measures are mutually singular, and the total variation of u,

$$|Du|(\Omega) := \sup\left\{\int_\Omega u \operatorname{div}\phi(x)\,\mathrm{d}x\colon\ \phi \in C_c^1(\Omega;\mathbb{R}^N),\ \|\phi\|_\infty \leqslant 1\right\},$$

is now

$$|Du|(\Omega) = \int_\Omega |\nabla u|\,\mathrm{d}x + \int_{S(u)} |u^+ - u^-|\,\mathrm{d}\mathcal{H}^{N-1} + |C(u)|(\Omega).$$

We recall that if $\{u_n\} \subset BV(\Omega;\mathbb{R}^d)$ and $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, then

$$|Du|(\Omega) \leqslant \liminf_{n\to\infty} |Du_n|(\Omega).$$

We say that a set $E \subset \Omega$ is of *finite perimeter* if $\chi_E \in BV(\Omega;\mathbb{R})$, and we denote by $\mathrm{Per}_\Omega(E)$ the perimeter of E in Ω, i.e., $\mathrm{Per}_\Omega(E) := |D\chi_E|(\Omega)$ given by

$$\mathrm{Per}_\Omega(E) := \sup\left\{\int_E \operatorname{div}\phi(x)\,\mathrm{d}x\colon\ \phi \in C_c^1(\Omega;\mathbb{R}^N),\ \|\phi\|_\infty \leqslant 1\right\}. \tag{2.1}$$

Definition 2.1. Let $A \subset \mathbb{R}^N$ be a set of locally finite perimeter and let $x_0 \in \mathbb{R}^N$. We say that x_0 belongs to the reduced boundary of A (and we write $x_0 \in \partial^* A$) if , with $D\chi_A = -\nu|D\chi_A|$, we have

(i) $|D\chi_A|(B(x_0,\varepsilon)) > 0$ for all $\varepsilon > 0$;
(ii) $\lim_{\varepsilon\to 0^+} \dfrac{1}{\mathcal{L}^N(B(x_0,\varepsilon))}\displaystyle\int_{B(x_0,\varepsilon)} \nu(x)\,\mathrm{d}|D\chi_A|(x) = \nu(x_0)$;
(iii) $\|\nu(x_0)\| = 1$.

ν is said to be *the outward unit normal to the boundary of A at x.*

Theorem 2.2 (see [15,20]). *If $x \in \partial^* A$ then*

$$\lim_{\delta\to 0^+} \frac{1}{\delta^N}\mathcal{L}^N\big(\{y \in B(x,\delta)\setminus A\colon\ (y-x)\cdot\nu(x) < 0\}\big) = 0,$$

$$\lim_{\delta\to 0^+} \frac{1}{\delta^N}\mathcal{L}^N\big(\{y \in B(x,\delta)\cap A\colon\ (y-x)\cdot\nu(x) > 0\}\big) = 0.$$

It can be shown (see [18]) that if $\mathrm{Per}_\Omega(A) < \infty$ then for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap \partial^* A$

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{N-1}}\mathcal{H}^{N-1}\big((\Omega\cap\partial^* A)\cap(x+\delta Q_{\nu(x)})\big) = 1. \tag{2.2}$$

Theorem 2.3 (see [5, Lemma 3.1]). *Let A be a subset of Ω such that* $\mathrm{Per}_\Omega(A) < \infty$. *There exists a sequence of polyhedral sets $\{A_k\}$ (i.e., A_k are bounded, Lipschitz domains with $\partial A_k = H_1 \cup H_2 \cup \cdots \cup H_p$, where each H_i is a closed subset of a hyperplane $\{x \in \mathbb{R}^N\colon\ x\cdot\nu_i = \alpha_i\}$) satisfying the following properties*:

(i) $\lim_{k\to\infty} \mathcal{L}^N[((A_k \cap \Omega) \setminus A) \cup (A \setminus (A_k \cap \Omega))] = 0$;
(ii) $\lim_{k\to\infty} \mathrm{Per}_\Omega(A_k) = \mathrm{Per}_\Omega(A)$;
(iii) $\mathcal{H}^{N-1}(\partial A_k \cap \partial\Omega) = 0$;
(iv) $\mathcal{L}^N(A_k) = \mathcal{L}^N(A)$.

Let $\varepsilon_n \to 0^+$. A functional

$$I : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$$

is called the Γ-lim inf (resp. Γ-lim sup) of a sequence of functionals $\{I_{\varepsilon_n}\}$ with respect to the strong convergence in $L^1(\Omega;\mathbb{R}^d)$ if for every $u \in L^1(\Omega;\mathbb{R}^d)$

$$I(u) = \inf\Big\{\liminf_{n\to\infty} (\text{resp. } \limsup_{n\to\infty}) I_{\varepsilon_n}(u_n)\text{: } u_n \in L^1(\Omega;\mathbb{R}^d),\ u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d)\Big\},$$

and we write

$$I = \Gamma\text{-}\liminf_{n\to\infty} I_{\varepsilon_n} \Big(\text{resp. } I = \Gamma\text{-}\limsup_{n\to\infty} I_{\varepsilon_n}\Big).$$

We say that the sequence $\{I_{\varepsilon_n}\}$ *Γ-converges* to I if the Γ-lim inf and the Γ-lim sup coincide, and we write

$$I = \Gamma\text{-}\lim_{n\to\infty} I_{\varepsilon_n}.$$

The functional I is said to be the Γ-lim inf (resp. Γ-lim sup) of the family of functionals $\{I_\varepsilon\}$ with respect to the strong convergence in $L^1(\Omega;\mathbb{R}^d)$ if for every sequence $\varepsilon_n \to 0^+$ we have that

$$I = \Gamma\text{-}\liminf_{n\to\infty} I_{\varepsilon_n} \Big(\text{resp. } I = \Gamma\text{-}\limsup_{n\to\infty} I_{\varepsilon_n}\Big),$$

and we write

$$I = \Gamma\text{-}\liminf_{\varepsilon\to 0} I_\varepsilon \Big(\text{resp. } I = \Gamma\text{-}\limsup_{\varepsilon\to 0} I_\varepsilon\Big).$$

Finally, if Γ-lim inf and Γ-lim sup coincide, we say that I is the *Γ-limit* of the family of functionals $\{I_\varepsilon\}$, and we write

$$I = \Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon.$$

In order to prove Theorem 1.1, it is enough to show that every sequence $\{\varepsilon_n\}$ of positive numbers converging to zero has a subsequence $\{\varepsilon_{n_k}\}$ such that $J_{\varepsilon_{n_k}}\,\Gamma(L^1(\Omega;\mathbb{R}^d))$-converges to J_0 (see [11,12]). We divide the proof of Theorem 1.1 into two parts. The first part is dealt with in Section 3 and the second part is left for Section 4 of the paper.

In the sequel, C will denote a generic positive constant that may vary from expression to expression.

3. A lower bound for the Γ-limit

In this section we prove

Proposition 3.1. *Let* (H1)–(H3) *hold and let* $u \in L^1(\Omega;\mathbb{R}^d)$ *be given. If* $\varepsilon_n \to 0^+$ *and if* $\{u_n\} \subset H^1(\Omega;\mathbb{R}^d)$ *is such that* $u_n \to u$ *in* $L^1(\Omega;\mathbb{R}^d)$*, then*

$$\liminf_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x \geqslant J_0(u).$$

The proof relies on the following lemma:

Lemma 3.2. *Assume that* (H1)–(H3) *hold, let* ν *be a unit vector and let*

$$u_0(x) := \begin{cases} b & \text{if } x\cdot\nu > 0,\\ a & \text{if } x\cdot\nu < 0.\end{cases}$$

Let $\rho:\mathbb{R}\to[0,+\infty)$ *be a symmetric mollifier and set* $v_n := \rho_{\frac{1}{\varepsilon_n},\nu} * u_0$*, where* $\rho_{\frac{1}{\varepsilon_n},\nu}(x) := (\frac{1}{\varepsilon_n})^N \rho(\frac{x\cdot\nu}{\varepsilon_n})$ *and* $\{\varepsilon_n\}$ *is a sequence of real numbers such that* $\varepsilon_n \to 0^+$.

If $\{u_n\}$ *is a sequence in* $H^1(Q_\nu;\mathbb{R}^d)$ *converging in* $L^1(Q_\nu;\mathbb{R}^d)$ *to* u_0*, then there exists a sequence* $\{w_n\}$ *in* $H^1(Q_\nu;\mathbb{R}^d)$ *such that* $w_n \to u_0$ *in* $L^1(Q_\nu;\mathbb{R}^d)$*,* $w_n = v_n$ *on* ∂Q_ν*, and*

$$\limsup_{n\to+\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f\big(x, w_n(x), \varepsilon_n \nabla w_n(x)\big)\,\mathrm{d}x \leqslant \liminf_{n\to+\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x.$$

Proof. *Step 1.* Assume, without loss of generality, that

$$\liminf_{n\to+\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x = \lim_{n\to+\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x < +\infty, \tag{3.1}$$

and that $u_n(x) \to u_0(x)$ $\mathcal{L}^N$-a.e. $x \in Q_\nu$. By (3.1)

$$\int_{Q_\nu} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x = \varepsilon_n \int_{Q_\nu} \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty. \tag{3.2}$$

By (H2) we have that

$$\big|u_n(x) - u_0(x)\big|^q \leqslant C\big(f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big) + 1\big),$$

and by Fatou's Lemma and (3.2)

$$\begin{aligned} C|Q_\nu| - \limsup_{n\to+\infty} \int_{Q_\nu} |u_n - u_0|^q\,\mathrm{d}x &= \liminf_{n\to+\infty} \int_{Q_\nu} \big(Cf\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big) + C - |u_n - u_0|^q\big)\,\mathrm{d}x \\ &\geqslant \int_{Q_\nu} \liminf_{n\to+\infty} \big[Cf\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big) + C - |u_n - u_0|^q\big]\,\mathrm{d}x \\ &\geqslant C|Q_\nu|. \end{aligned}$$

Therefore,

$$\limsup_{n\to+\infty}\int_{Q_\nu}|u_n-u_0|^q\,\mathrm{d}x=0, \tag{3.3}$$

and in particular, since $q\geqslant 2$, we conclude that $u_n\to u_0$ in $L^2(Q_\nu;\mathbb{R}^d)$ as $n\to+\infty$.

Step 2. For simplicity, assume that $\nu=e_N$ and denote $Q_\nu=Q$. Notice that

$$v_n(x)=\begin{cases} b & \text{if } x_N>\varepsilon_n,\\ a & \text{if } x_N<-\varepsilon_n,\end{cases}$$

and

$$\begin{aligned}&v_n\in\mathcal{A}(a,b,e_N),\qquad \|\nabla v_n\|_\infty=\mathrm{O}(1/\varepsilon_n),\qquad \operatorname{supp}\nabla v_n\subset\{x\in Q\colon\ |x_N|<\varepsilon_n\},\quad\text{and}\\ &\quad v_n\to u_0\quad\text{in } L^q(Q;\mathbb{R}^d).\end{aligned} \tag{3.4}$$

For each $k\in\mathbb{N}$ define

$$L_k:=\left\{x\in Q\colon\ \mathrm{dist}(x,\partial Q)\leqslant\frac{1}{k}\right\}.$$

Consider n sufficiently large ($n\geqslant n(k)$ for some $n(k)$), and divide L_k into $M_{k,n}$ layers $L_{k,n}^{(i)}$ ($i=1,\dots,M_{k,n}$) of width $\varepsilon_n\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}^{1/2}$, so that $M_{k,n}\varepsilon_n\|u_n-v_n\|_2^{1/2}=\mathrm{O}(1/k)$.

We have that

$$\begin{aligned}&\sum_{i=1}^{M_{k,n}}\int_{L_{k,n}^{(i)}}\left(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2+\varepsilon_n^2|\nabla v_n|^2+\frac{|u_n-v_n|^2}{\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}}\right)\mathrm{d}x\\ &\quad=\int_{L_k}\left(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2+\varepsilon_n^2|\nabla v_n|^2+\frac{|u_n-v_n|^2}{\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}}\right)\mathrm{d}x,\end{aligned}$$

and thus there exists $i=i(k,n)\in\{1,\dots,M_{k,n}\}$ such that

$$\begin{aligned}&\int_{L_{k,n}^{(i)}}\left(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2+\varepsilon_n^2|\nabla v_n|^2+\frac{|u_n-v_n|^2}{\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}}\right)\mathrm{d}x\\ &\quad\leqslant\frac{1}{M_{k,n}}\int_{L_k}\left(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2+\varepsilon_n^2|\nabla v_n|^2+\frac{|u_n-v_n|^2}{\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}}\right)\mathrm{d}x.\end{aligned} \tag{3.5}$$

Choose cut-off functions $\varphi_{k,n}\in C_c^\infty(Q;[0,1])$ such that $\varphi_{k,n}=0$ on $\bigcup_{j=i+1}^{M_{k,n}}L_{k,n}^{(j)}=:A_{k,n}$, $\varphi_{k,n}=1$ on $(Q\setminus L_k)\cup(\bigcup_{j=1}^{i-1}L_{k,n}^{(j)})=:B_{k,n}$. Define

$$w_{k,n}:=\varphi_{k,n}u_n+(1-\varphi_{k,n})v_n.$$

We have that

$$\lim_{k\to\infty}\lim_{n\to\infty}\|w_{k,n}-u_0\|_{L^1(Q;\mathbb{R}^d)}=0.$$

Also

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q\frac{1}{\varepsilon_n}f\bigl(x,w_{k,n}(x),\varepsilon_n\nabla w_{k,n}(x)\bigr)\,\mathrm{d}x\\
&\quad\leqslant\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{A_{k,n}}\frac{1}{\varepsilon_n}f\bigl(x,v_n(x),\varepsilon_n\nabla v_n(x)\bigr)\,\mathrm{d}x\\
&\qquad+\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{L^{(i)}_{k,n}}\frac{1}{\varepsilon_n}f\bigl(x,w_{k,n}(x),\varepsilon_n\nabla w_{k,n}(x)\bigr)\,\mathrm{d}x\\
&\qquad+\lim_{n\to\infty}\int_Q\frac{1}{\varepsilon_n}f\bigl(x,u_n(x),\varepsilon_n\nabla u_n(x)\bigr)\,\mathrm{d}x. \qquad (3.6)
\end{aligned}$$

By (H2) and (3.4) we have

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{A_{k,n}}\frac{1}{\varepsilon_n}f\bigl(x,v_n(x),\varepsilon_n\nabla v_n(x)\bigr)\,\mathrm{d}x\\
&\quad\leqslant\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{L_k\cap\{x\in Q:\ |x_N|<\varepsilon_n\}}\frac{C}{\varepsilon_n}\bigl(g(x,v_n)+\varepsilon_n^2|\nabla v_n|^2\bigr)\,\mathrm{d}x\\
&\quad\leqslant\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{L_k\cap\{x\in Q:\ |x_N|<\varepsilon_n\}}\frac{C}{\varepsilon_n}\bigl(1+|v_n|^q+\varepsilon_n^2|\nabla v_n|^2\bigr)\,\mathrm{d}x=0.
\end{aligned}$$

Also, in view of (3.5) and (H2),

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{L^{(i)}_{k,n}}\frac{1}{\varepsilon_n}f\bigl(x,w_{k,n}(x),\varepsilon_n\nabla w_{k,n}(x)\bigr)\,\mathrm{d}x\\
&\quad\leqslant\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{C}{\varepsilon_n M_{k,n}}\int_{L_k}\biggl(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2+\varepsilon_n^2|\nabla v_n|^2\\
&\qquad+\frac{|u_n-v_n|^2}{\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}}\biggr)\,\mathrm{d}x\\
&\quad\leqslant\limsup_{k\to\infty}\limsup_{n\to\infty}Ck\|u_n-v_n\|_2^{1/2}\biggl(\int_Q\bigl(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2+\varepsilon_n^2|\nabla v_n|^2\bigr)\,\mathrm{d}x\\
&\qquad+\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}\biggr)\\
&\quad=0,
\end{aligned}$$

where we have used (3.3). Thus, (3.6) becomes

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q\frac{1}{\varepsilon_n}f\bigl(x,w_{k,n}(x),\varepsilon_n\nabla w_{k,n}(x)\bigr)\,\mathrm{d}x\leqslant\lim_{n\to\infty}\int_Q\frac{1}{\varepsilon_n}f\bigl(x,u_n(x),\varepsilon_n\nabla u_n(x)\bigr)\,\mathrm{d}x.$$

Using a diagonalization process (see Lemma 7.1 in [9]) we extract a subsequence $\{k(n)\}$ of $\{k\}$ such that, letting $w_n := w_{k(n),n}$, we have

$$\lim_{n\to\infty} \|w_n - u_0\|_{L^1} = 0,$$

$$\limsup_{n\to\infty} \int_Q \frac{1}{\varepsilon_n} f\big(x, w_n(x), \varepsilon_n \nabla w_n(x)\big)\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_Q \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x. \qquad \square$$

Proof of Proposition 3.1. *Step 1.* If $u \in L^1(\Omega;\mathbb{R}^d)$ and $\mathcal{L}^N(\{x \in \Omega\colon\ u(x) \notin \{a,b\}\}) > 0$ then for any sequence $\varepsilon_n \to 0^+$ and for any $\{u_n\} \subset H^1(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, we have

$$\int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x \to +\infty. \tag{3.7}$$

Indeed, if for some sequences $\varepsilon_n \to 0^+$, and $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$

$$\sup_{n\in\mathbb{N}} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x < +\infty,$$

then by (H2) and Fatou's lemma $\int_\Omega g(x,u)\,\mathrm{d}x = 0$, and thus $u(x) \in \{a,b\}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, that is a contradiction.

Step 2. Consider now the case where $u = \chi_{A_0}(x)a + (1 - \chi_{A_0}(x))b$ and $u \notin BV(\Omega;\mathbb{R}^d)$, i.e., $\mathrm{Per}_\Omega(A_0) = +\infty$. Again, we show that (3.7) is satisfied. We argue by contradiction. Suppose that there exists a subsequence (not relabeled) such that

$$\sup_{n\in\mathbb{N}} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x \leqslant C.$$

Then, by (H2), we have

$$\sup_{n\in\mathbb{N}} \int_\Omega \left[\frac{1}{\varepsilon_n} g\big(x, u_n(x)\big) + \varepsilon_n \big|\nabla u_n(x)\big|^2\right] \mathrm{d}x \leqslant C,$$

which, by the Cauchy–Schwarz inequality, implies that

$$\sup_{n\in\mathbb{N}} \int_\Omega \sqrt{g\big(x, u_n(x)\big)}\,\big|\nabla u_n(x)\big|\,\mathrm{d}x \leqslant C. \tag{3.8}$$

Setting $\bar{g}(u) := \min_{x\in\overline{\Omega}} g(x,u)$, we note that $\bar{g}(u) = 0$ if and only if $u \in \{a,b\}$, with $\bar{g}(u) \geqslant C|u|$ for suitable $C > 0$ and $|u|$ sufficiently large. In view of Lemma 3.7 in [19], for suitable $M > 0$ the function

$$\Phi(u) := \inf\left\{\int_{-1}^{1} \sqrt{\min\{\bar{g}(\gamma(s)), M\}}\,\big|\gamma'(s)\big|\,\mathrm{d}s\colon\ \gamma \text{ is continuous and piecewise } C^1,\right.$$
$$\left.\gamma(-1) = a,\ \gamma(1) = u\right\}$$

is Lipschitz continuous and $|\nabla(\Phi \circ v)(x)| \leqslant \sqrt{\bar{g}(v(x))}|\nabla v(x)|$ for any $v \in H^1(\Omega;\mathbb{R}^d)$, and $\mathcal{L}^N$-a.e. $x \in \Omega$. Thus

$$\sup_{n\in\mathbb{N}} \|\nabla(\Phi \circ u_n)\|_{L^1(\Omega;\mathbb{R}^d)} < +\infty.$$

Therefore $|D(\Phi\circ u)|(\Omega) < +\infty$, and since $\Phi\circ u = (1-\chi_{A_0})\Phi(b)$ we obtain that $\mathrm{Per}_\Omega(A_0) < +\infty$. Here we are using the fact that $\Phi(b) > 0$. Indeed, if γ is admissible for $\Phi(b)$, by continuity find $t_1 \in (-1,1)$ such that

$$\bigl|\gamma(t_1) - b\bigr| = \frac{|a-b|}{3} \quad \text{and} \quad \bigl|\gamma(s) - b\bigr| > \frac{|a-b|}{3} \quad \text{for all } -1 \leqslant s < t_1. \tag{3.9}$$

Let $t_0 := \max\{t \in (-1,t_1)\colon\ |\gamma(t) - a| = \frac{|a-b|}{3}\}$. By the Mean Value Theorem it follows that

$$\bigl|\gamma(t) - a\bigr| > \frac{|a-b|}{3} \quad \text{for all } s \in (t_0,t_1). \tag{3.10}$$

Set

$$\alpha := \min\biggl\{\sqrt{\min\{\bar{g}(x),M\}}\colon\ |x-a| \geqslant \frac{|a-b|}{3} \text{ and } |x-b| \geqslant \frac{|a-b|}{3}\biggr\}.$$

Then $\alpha > 0$, and in view of (3.9) and (3.10) we have

$$\begin{aligned}\int_{-1}^{1} \sqrt{\min\{\bar{g}(\gamma(s)),M\}}\,\bigl|\gamma'(s)\bigr|\,\mathrm{d}s &\geqslant \alpha\int_{t_0}^{t_1} \bigl|\gamma'(s)\bigr|\,\mathrm{d}s \geqslant \alpha\biggl|\int_{t_0}^{t_1} \gamma'(s)\,\mathrm{d}s\biggr| \\ &= \alpha\bigl|\gamma(t_1) - \gamma(t_0)\bigr| \geqslant \alpha\frac{|a-b|}{3}.\end{aligned}$$

We conclude that $\Phi(b) \geqslant \alpha\frac{|a-b|}{3}$.

Step 3. We look now at the case where $u = \chi_{A_0}(x)a + (1-\chi_{A_0}(x))b$ with $\mathrm{Per}_\Omega(A_0) < +\infty$. Assume, without loss of generality, that

$$\liminf_{n\to\infty}\int_\Omega \frac{1}{\varepsilon_n} f\bigl(x,u_n(x),\varepsilon_n\nabla u_n(x)\bigr)\,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega \frac{1}{\varepsilon_n} f\bigl(x,u_n(x),\varepsilon_n\nabla u_n(x)\bigr)\,\mathrm{d}x < +\infty.$$

We must show that

$$\lim_{n\to\infty}\int_\Omega \frac{1}{\varepsilon_n} f\bigl(x,u_n(x),\varepsilon_n\nabla u_n(x)\bigr)\,\mathrm{d}x \geqslant \int_{\Omega\cap\partial^* A_0} K\bigl(x,a,b,\nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x). \tag{3.11}$$

Since the integrands $\frac{1}{\varepsilon_n} f(x,u_n(x),\varepsilon_n\nabla u_n(x))$ form a sequence of non-negative functions bounded in $L^1(\Omega;\mathbb{R}^d)$, there exists a subsequence (not relabeled) and a non-negative Radon measure μ such that

$$\frac{1}{\varepsilon_n} f\bigl(\cdot,u_n(\cdot),\varepsilon_n\nabla u_n(\cdot)\bigr) \rightharpoonup \mu \quad \text{weakly} * \text{ in the sense of measures.} \tag{3.12}$$

Using the Radon–Nikodym Theorem, we may write μ as a sum of two mutually singular non-negative measures $\mu = \mu_a \mathcal{H}^{N-1} \lfloor (\Omega \cap \partial^* A_0) + \mu_s$, where μ_a is a suitable Borel function on $\Omega \cap \partial^* A_0$. We claim that

$$\mu_a(x) \geqslant K\bigl(x, a, b, \nu(x)\bigr) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega \cap \partial^* A_0. \tag{3.13}$$

Assuming that (3.13) holds, we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \frac{1}{\varepsilon_n} f\bigl(x, u_n(x), \varepsilon_n \nabla u_n(x)\bigr)\,\mathrm{d}x \geqslant \mu(\Omega) &\geqslant \int_\Omega \mu_a(x)\,\mathrm{d}\mathcal{H}^{N-1}\lfloor(\Omega \cap \partial^* A_0)(x)\\ &\geqslant \int_{\Omega\cap\partial^* A_0} K\bigl(x, a, b, \nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x),\end{aligned}$$

and we deduce (3.11). It remains to show that (3.13) holds.

By Theorem 2.2 for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap \partial^* A_0$ we have

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N} \mathcal{L}^N\bigl(\{y \in Q(x,\rho) \cap A_0\colon (y-x)\cdot\nu(x) > 0\}\bigr) = 0, \tag{3.14}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N} \mathcal{L}^N\bigl(\{y \in Q(x,\rho) \setminus A_0\colon (y-x)\cdot\nu(x) < 0\}\bigr) = 0, \tag{3.15}$$

$$\mu_a(x) = \lim_{\rho\to 0^+} \frac{\mu(x+\rho Q_{\nu(x)})}{\mathcal{H}^{N-1}\lfloor(\Omega\cap\partial^* A_0)(x+\rho Q_{\nu(x)})},$$

and by (2.2), (3.12), and choosing $\rho_k \to 0^+$ such that $\mu(\partial(x+\rho_k Q_{\nu(x)})) = 0$, we have

$$\begin{aligned}\mu_a(x) &= \lim_{\rho\to 0^+} \frac{\mu(x+\rho Q_{\nu(x)})}{\rho^{N-1}}\\ &= \lim_{k\to\infty} \frac{1}{\rho_k^{N-1}} \lim_{n\to\infty} \int_{x+\rho_k Q_{\nu(x)}} \frac{1}{\varepsilon_n} f\bigl(y, u_n(y), \varepsilon_n \nabla u_n(y)\bigr)\,\mathrm{d}y\\ &= \lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} f\bigl(x+\rho_k y, u_n(x+\rho_k y), \varepsilon_n \nabla u_n(x+\rho_k y)\bigr)\,\mathrm{d}y.\end{aligned} \tag{3.16}$$

Let

$$w_{n,k}(y) := u_n(x+\rho_k y), \qquad u_0(y) := \begin{cases} b & \text{if } y\cdot\nu(x) > 0,\\ a & \text{if } y\cdot\nu(x) < 0.\end{cases}$$

Since $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, and in view of (3.14) and (3.15), we have that

$$\begin{aligned}&\lim_{k\to\infty}\lim_{n\to\infty} \|w_{n,k} - u_0\|_{L^1(Q_{\nu(x)};\mathbb{R}^d)}\\ &\quad= \lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_{\nu(x)}} \bigl|w_{n,k}(y) - u_0(y)\bigr|\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\lim_{n\to\infty}\biggl[\int_{Q_{\nu(x)}\cap\{y:\ y\cdot\nu(x)>0\}} \bigl|u_n(x+\rho_k y)-b\bigr|\,\mathrm{d}y \\
&\quad + \int_{Q_{\nu(x)}\cap\{y:\ y\cdot\nu(x)<0\}} \bigl|u_n(x+\rho_k y)-a\bigr|\,\mathrm{d}y\biggr] \\
&= \lim_{k\to\infty}\biggl[\int_{Q_{\nu(x)}\cap\{y:\ y\cdot\nu(x)>0\}} \bigl|u(x+\rho_k y)-b\bigr|\,\mathrm{d}y + \int_{Q_{\nu(x)}\cap\{y:\ y\cdot\nu(x)<0\}} \bigl|u(x+\rho_k y)-a\bigr|\,\mathrm{d}y\biggr] \\
&= \lim_{k\to\infty}\biggl[\frac{1}{\rho_k^N}\int_{(x+\rho_k Q_{\nu(x)})\cap\{y:\ (y-x)\cdot\nu(x)>0\}\cap A_0} |a-b|\,\mathrm{d}y \\
&\quad + \frac{1}{\rho_k^N}\int_{(x+\rho_k Q_{\nu(x)})\cap\{y:\ (y-x)\cdot\nu(x)<0\}\setminus A_0} |b-a|\,\mathrm{d}y\biggr] \\
&= 0.
\end{aligned}
$$

Since $\nabla w_{n,k}(y)=\rho_k\nabla u_n(x+\rho_k y)$, (3.16) yields

$$
\mu_a(x) = \lim_{k\to\infty}\lim_{n\to+\infty}\int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} f\biggl(x+\rho_k y, w_{n,k}(y), \frac{\varepsilon_n}{\rho_k}\nabla w_{n,k}(y)\biggr)\,\mathrm{d}y. \tag{3.17}
$$

Choose $n_k\in\mathbb{N}$ large enough so that $\frac{\varepsilon_{n_k}}{\rho_k}\to 0$, $\lim_{k\to\infty}\|w_{n_k,k}-u_0\|_{L^1(Q_{\nu(x)};\mathbb{R}^d)}=0$, and

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_{n_k}} f\biggl(x+\rho_k y, w_{n_k,k}(y), \frac{\varepsilon_{n_k}}{\rho_k}\nabla w_{n_k,k}(y)\biggr)\,\mathrm{d}y \\
&\quad = \lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} f\biggl(x+\rho_k y, w_{n,k}(y), \frac{\varepsilon_n}{\rho_k}\nabla w_{n,k}(y)\biggr)\,\mathrm{d}y.
\end{aligned} \tag{3.18}
$$

Applying Lemma 3.2 to the sequences $\{w_{n_k,k}\}$ and $\{\alpha_k\} := \{\frac{\varepsilon_{n_k}}{\rho_k}\}$, we conclude that there exists a sequence $\{\xi_k\}\subset H^1(Q_{\nu(x)};\mathbb{R}^d)$ such that $\xi_k\to u_0$ in $L^1(Q_{\nu(x)};\mathbb{R}^d)$, $\xi_k\in\mathcal{A}(a,b,\nu(x))$, and

$$
\begin{aligned}
&\limsup_{k\to\infty}\int_{Q_{\nu(x)}} \frac{1}{\alpha_k} f\bigl(x,\xi_k(z),\alpha_k\nabla\xi_k(z)\bigr)\,\mathrm{d}z \\
&\quad \leqslant \lim_{k\to\infty}\int_{Q_{\nu(x)}} \frac{1}{\alpha_k} f\bigl(x,w_{n_k,k}(z),\alpha_k\nabla w_{n_k,k}(z)\bigr)\,\mathrm{d}z.
\end{aligned} \tag{3.19}
$$

Thus, by (3.17), (3.18), (3.19), and taking into account hypothesis (H3), we have

$$
\begin{aligned}
\mu_a(x) &\geqslant \lim_{k\to\infty}\int_{Q_{\nu(x)}} \frac{1}{\alpha_k} f\bigl(x,w_{n_k,k}(z),\alpha_k\nabla w_{n_k,k}(z)\bigr)\,\mathrm{d}z \\
&\quad - \limsup_{k\to\infty}\int_{Q_{\nu(x)}} \frac{1}{\alpha_k}\bigl[f\bigl(x,w_{n_k,k}(y),\alpha_k\nabla w_{n_k,k}(y)\bigr) \\
&\quad - f\bigl(x+\rho_k y,w_{n_k,k}(y),\alpha_k\nabla w_{n_k,k}(y)\bigr)\bigr]\,\mathrm{d}y \\
&\geqslant K\bigl(x,a,b,\nu(x)\bigr).
\end{aligned}
$$

4. An upper bound for the Γ-limit

In this section we show that $\Gamma\text{-}\limsup_{\varepsilon\to 0^+} J_\varepsilon \leqslant J_0$. In view of Steps 1 and 2 in the proof of Proposition 3.1, it suffices to prove

Proposition 4.1. *Assume that the hypotheses* (H1)–(H3) *hold. Given any* $u \in BV(\Omega;\{a,b\})$ *and any sequence* $\varepsilon_n \to 0^+$, *there exists a sequence* $\{u_n\} \subset H^1(\Omega;\mathbb{R}^d)$ *such that* $u_n \to u$ *in* $L^1(\Omega;\mathbb{R}^d)$ *and*

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x)\big)\,\mathrm{d}x = J_0(u). \tag{4.1}$$

We will achieve this by showing that given any sequence $\varepsilon_n \to 0^+$, (4.1) holds for a subsequence $\{\varepsilon_n^{\mathcal{R}}\}$ of $\{\varepsilon_n\}$. Indeed, recalling the main result of the previous section (Proposition 3.1) we then obtain that the $\Gamma(L^1)$-limit of $J_{\varepsilon_n^{\mathcal{R}}}(u)$ is $J_0(u)$, which is independent on the specific subsequence $\{\varepsilon_n^{\mathcal{R}}\}$. In light of Proposition 7.11 in [8], we deduce that, in fact, $J_\varepsilon(u)$ $\Gamma(L^1)$-converges to $J_0(u)$.

We begin by considering the particular case where $u := \chi_{A_0}(x)a + (1-\chi_{A_0}(x))b$ has planar interface and f and K do not depend explicitly on x.

Lemma 4.2. *Assume that* (H1)–(H3) *hold,* $\eta > 0$, *let* u_0 *be as in Lemma* 3.2, *and*

$$u(x) := \begin{cases} b & \text{if } (x-a_0)\cdot\nu > 0,\\ a & \text{if } (x-a_0)\cdot\nu < 0,\end{cases}$$

for some $a_0 \in \mathbb{R}^N$. *Assume that f does not depend on x. Then, for every sequence* $\varepsilon_n \to 0^+$, *there exists a sequence* $\{u_n\} \subset H^1(a_0+\eta Q_\nu;\mathbb{R}^d)$ *such that* $u_n \to u$ *in* $L^1(a_0+\eta Q_\nu;\mathbb{R}^d)$, *and*

$$\lim_{n\to+\infty}\int_{a_0+\eta Q_\nu} \frac{1}{\varepsilon_n} f\big(u_n(x), \varepsilon_n\nabla u_n(x)\big)\,\mathrm{d}x = \eta^{N-1}K(a,b,\nu) = J_0(u).$$

Proof. For simplicity, we assume that $\nu = e_N$ and we denote Q_ν by Q. Let Q' be the projection of Q on $\mathbb{R}^{N-1}$, $Q' := \{x \in Q\colon x_N = 0\}$.

Case 1. Suppose first that $a_0 = 0$ and $\eta = 1$. Let $L_n > 0$ and $\xi_n \in \mathcal{A}(a,b,e_N)$ be such that

$$\lim_{n\to\infty}\int_Q \frac{1}{L_n} f\big(\xi_n(x), L_n\nabla\xi_n(x)\big)\,\mathrm{d}x = K(a,b,e_N). \tag{4.2}$$

Define

$$v_n^k(x) := \begin{cases} b & \text{if } x_N > \dfrac{\varepsilon_n L_k}{2},\\[2mm] \xi_k\Big(\dfrac{x}{\varepsilon_n L_k}\Big) & \text{if } |x_N| \leqslant \dfrac{\varepsilon_n L_k}{2},\\[2mm] a & \text{if } x_N < -\dfrac{\varepsilon_n L_k}{2}.\end{cases}$$

Clearly, $v_n^k \in \mathcal{A}(a,b,e_N)$ for all $k,n \in \mathbb{N}$, and $v_n^k \to u$ in $L^1(Q;\mathbb{R}^d)$ as $n \to \infty$. On the other hand,

$$\begin{aligned}
&\int_Q \frac{1}{\varepsilon_n} f\big(v_n^k(x), \varepsilon_n \nabla v_n^k(x)\big)\,\mathrm{d}x \\
&\quad = \int_{Q\cap\{x:\ |x_N|\leqslant\frac{\varepsilon_n L_k}{2}\}} \frac{1}{\varepsilon_n} f\left(\xi_k\left(\frac{x'}{\varepsilon_n L_k}, \frac{x_N}{\varepsilon_n L_k}\right), \frac{1}{L_k}\nabla\xi_k\left(\frac{x'}{\varepsilon_n L_k}, \frac{x_N}{\varepsilon_n L_k}\right)\right)\mathrm{d}x \\
&\quad = \int_{-1/2}^{1/2}\int_{Q'} L_k f\left(\xi_k\left(\frac{x'}{\varepsilon_n L_k}, x_N\right), \frac{1}{L_k}\nabla\xi_k\left(\frac{x'}{\varepsilon_n L_k}, x_N\right)\right)\mathrm{d}x'\,\mathrm{d}x_N.
\end{aligned}$$

Therefore, in view of the Riemann–Lebesgue Lemma, we obtain by (4.2)

$$\begin{aligned}
\lim_{k\to\infty}\lim_{n\to\infty}\int_Q \frac{1}{\varepsilon_n} f\big(v_n^k(x), \varepsilon_n \nabla v_n^k(x)\big)\,\mathrm{d}x &= \lim_{k\to\infty}\int_Q L_k f\left(\xi_k(x), \frac{1}{L_k}\nabla\xi_k(x)\right)\mathrm{d}x \\
&= K(a,b,e_N).
\end{aligned}$$

Since we also have $\lim_{k\to\infty}\lim_{n\to\infty}\|v_n^k - u\|_{L^1} = 0$, a diagonalization argument (see Lemma 7.1 in [9]) produces a subsequence $\{k(n)\} \subset \{k\}$ such that

$$\lim_{n\to\infty}\int_Q \frac{1}{\varepsilon_n} f\big(v_n^{k(n)}(x), \varepsilon_n \nabla v_n^{k(n)}(x)\big)\,\mathrm{d}x = K(a,b,e_N),$$

and

$$\lim_{n\to\infty}\big\|v_n^{k(n)} - u\big\|_{L^1} = 0.$$

It suffices to set $u_n := v_n^{k(n)}$.

Case 2. We now consider the general case where $a_0 \in \mathbb{R}^N$, and for $\eta > 0$ we define

$$f_\eta(u,A) := f\left(u, \frac{A}{\eta}\right).$$

Set

$$u_0(x) := \begin{cases} b & \text{if } x\cdot e_N > 0, \\ a & \text{if } x\cdot e_N < 0. \end{cases}$$

Given $\varepsilon_n \to 0^+$, by Case 1 we obtain a sequence $\{v_n\} \subset \mathcal{A}(a,b,e_N)$ such that $v_n \to u_0$ in $L^1(Q;\mathbb{R}^d)$, and

$$\lim_{n\to\infty}\int_Q \frac{1}{\varepsilon_n} f_\eta\big(v_n(x), \varepsilon_n \nabla v_n(x)\big)\,\mathrm{d}x = K_\eta(a,b,e_N),$$

where

$$K_\eta(a,b,e_N) := \inf_{L>0}\left\{\int_Q L f_\eta\left(\xi(x), \frac{1}{L}\nabla\xi(x)\right)\mathrm{d}x\colon\ \xi \in \mathcal{A}(a,b,e_N)\right\}.$$

Note that

$$K_\eta(a,b,e_N) = \frac{1}{\eta} K(a,b,e_N). \tag{4.3}$$

For $x \in a_0 + \eta Q$, define $u_n \in H^1(a_0 + \eta Q; \mathbb{R}^d)$ by

$$u_n(x) := v_n\left(\frac{x - a_0}{\eta}\right).$$

We have

$$\begin{aligned}\int_{a_0+\eta Q} \left|u_n(x) - u(x)\right| \mathrm{d}x &= \int_{a_0+\eta Q} \left|v_n\left(\frac{x-a_0}{\eta}\right) - u(x)\right| \mathrm{d}x = \eta^N \int_Q \left|v_n(x) - u(a_0 + \eta x)\right| \mathrm{d}x \\ &= \eta^N \int_Q \left|v_n(x) - u_0(x)\right| \mathrm{d}x \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Also, in view of (4.3),

$$\begin{aligned}&\lim_{n\to\infty} \int_{a_0+\eta Q} \frac{1}{\varepsilon_n} f\big(u_n(x), \varepsilon_n \nabla u_n(x)\big) \, \mathrm{d}x \\ &\quad= \lim_{n\to\infty} \int_{a_0+\eta Q} \frac{1}{\varepsilon_n} f\left(v_n\left(\frac{x-a_0}{\eta}\right), \frac{\varepsilon_n}{\eta} \nabla v_n\left(\frac{x-a_0}{\eta}\right)\right) \mathrm{d}x \\ &\quad= \lim_{n\to\infty} \eta^N \int_Q \frac{1}{\varepsilon_n} f\left(v_n(x), \frac{\varepsilon_n}{\eta} \nabla v_n(x)\right) \mathrm{d}x \\ &\quad= \lim_{n\to\infty} \eta^N \int_Q \frac{1}{\varepsilon_n} f_\eta\big(v_n(x), \varepsilon_n \nabla v_n(x)\big) \, \mathrm{d}x = \eta^N K_\eta(a,b,e_N) = \eta^{N-1} K(a,b,e_N). \qquad \square\end{aligned}$$

The last step of the proof of Proposition 4.1 uses the upper semicontinuity property of K. Precisely

Proposition 4.3. *If* (H2) *holds then*

(i) $0 \leqslant K(x,a,b,\nu) \leqslant C(1 + |a|^q + |b|^q)$ *for all* $(x,a,b,\nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$;
(ii) $K(\cdot,a,b,\cdot)$ *is upper semicontinuous.*

Proof. (i) Fix $(x,a,b,\nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$, and let

$$\xi(y) := (b-a)(y \cdot \nu) + \frac{a+b}{2}.$$

Since $\xi \in \mathcal{A}(a,b,\nu)$, and in view of (H2), we have

$$\begin{aligned}0 \leqslant K(x,a,b,\nu) &\leqslant \int_{Q_\nu} f\big(x, \xi(y), \nabla\xi(y)\big) \, \mathrm{d}y \leqslant \int_{Q_\nu} C\big(g\big(x,\xi(y)\big) + \left|\nabla\xi(y)\right|^2\big) \, \mathrm{d}y \\ &\leqslant \int_{Q_\nu} C\big(1 + \left|\xi(y)\right|^q + \left|\nabla\xi(y)\right|^2\big) \, \mathrm{d}y \leqslant C\big(1 + |a|^q + |b|^q\big).\end{aligned}$$

(ii) First, it is clear that

$$K(x,a,b,\nu) := \inf_{s>0}\left\{\frac{1}{s}\int_Q f\big(x,u(y),s\nabla u(y)R^{\mathrm{T}}\big)\,\mathrm{d}y\colon\ u\in\mathcal{A}(a,b,e_N),\ R\in SO(N),\ Re_N=\nu\right\}.$$

Let $(x_n,\nu_n)\to(x,\nu)$. Given $\varepsilon>0$ choose a rotation $R_{\varepsilon,x}$ such that $R_{\varepsilon,x}e_N=\nu,\ \xi_{\varepsilon,x}\in\mathcal{A}(a,b,e_N)$, and $s_{\varepsilon,x}>0$ such that

$$\left|K(x,a,b,\nu)-\int_Q\frac{1}{s_{\varepsilon,x}}f\big(x,\xi_{\varepsilon,x}(y),s_{\varepsilon,x}\nabla\xi_{\varepsilon,x}(y)R^{\mathrm{T}}_{\varepsilon,x}\big)\,\mathrm{d}y\right|<\varepsilon. \tag{4.4}$$

Let $R_n\in SO(N)$ be such that $R_ne_N=\nu_n$ ($n\in\mathbb{N}$), and $R_n\to R_{\varepsilon,x}$ as $n\to+\infty$. We have

$$K(x_n,a,b,\nu_n)\leqslant\int_Q\frac{1}{s_{\varepsilon,x}}f\big(x_n,\xi_{\varepsilon,x}(y),s_{\varepsilon,x}\nabla\xi_{\varepsilon,x}(y)R^{\mathrm{T}}_n\big)\,\mathrm{d}y. \tag{4.5}$$

Since f is continuous, and $\xi_{\varepsilon,x}\in\mathcal{A}(a,b,e_N)$, we have

$$\frac{1}{s_{\varepsilon,x}}f\big(x_n,\xi_{\varepsilon,x}(y),s_{\varepsilon,x}\nabla\xi_{\varepsilon,x}(y)R^{\mathrm{T}}_n\big)\to\frac{1}{s_{\varepsilon,x}}f\big(x,\xi_{\varepsilon,x}(y),s_{\varepsilon,x}\nabla\xi_{\varepsilon,x}(y)R^{\mathrm{T}}_{\varepsilon,x}\big),$$

pointwise, as $n\to\infty$, and

$$\begin{aligned}&\left|\frac{1}{s_{\varepsilon,x}}f\big(x_n,\xi_{\varepsilon,x}(y),s_{\varepsilon,x}\nabla\xi_{\varepsilon,x}(y)R^{\mathrm{T}}_n\big)\right|\\&\quad\leqslant\frac{1}{s_{\varepsilon,x}}\max\big\{f(z,u,\xi)\colon\ |z|\leqslant|x|+1,\ |u|\leqslant\|\xi_{\varepsilon,x}\|_\infty,|\xi|\leqslant\|\nabla\xi_{\varepsilon,x}\|_\infty\big\},\end{aligned}$$

for $y\in Q$, and $n\in\mathbb{N}$ sufficiently large. Thus, by Lebesgue's Dominated Convergence Theorem, we obtain that

$$\lim_{n\to\infty}\int_Q\frac{1}{s_{\varepsilon,x}}f\big(x_n,\xi_{\varepsilon,x}(y),s_{\varepsilon,x}\nabla\xi_{\varepsilon,x}(y)R^{\mathrm{T}}_n\big)\,\mathrm{d}y=\int_Q\frac{1}{s_{\varepsilon,x}}f\big(x,\xi_{\varepsilon,x}(y),s_{\varepsilon,x}\nabla\xi_{\varepsilon,x}(y)R^{\mathrm{T}}_{\varepsilon,x}\big)\,\mathrm{d}y.$$

Passing to lim sup as $n\to\infty$ in (4.5), and taking into account (4.4), we have

$$\limsup_{n\to\infty}K(x_n,a,b,\nu_n)\leqslant K(x,a,b,\nu)+\varepsilon.$$

We conclude the proof by letting $\varepsilon\to0^+$. □

Proof of Proposition 4.1. We divide the proof into two steps:

Step 1: A_0 is polyhedral. Let $\mathcal{C}$ be the family of all open cubes in Ω with faces parallel to the axes, centered at points $x\in\Omega\cap\mathbb{Q}^N$ and with rational edge length. Denote by $\mathcal{R}$ the countable subfamily of $\mathcal{A}(\Omega)$ obtained by taking all finite unions of elements of $\mathcal{C}$, i.e.,

$$\mathcal{R}:=\left\{\bigcup_{i=1}^{k}C_i\colon\ k\in\mathbb{N},\ C_i\in\mathcal{C}\right\}.$$

Let $\varepsilon_n \to 0^+$. Since $L^1(\Omega;\mathbb{R}^d)$ is a separable metric space, using Kuratowski's Compactness Theorem (see, e.g., [11]), a diagonalization argument, and in the spirit of Γ-convergence (see Proposition 7.9 in [8]), we can assert that there exists a subsequence $\{\varepsilon_n^{\mathcal{R}}\}$ of $\{\varepsilon_n\}$ such that, upon setting

$$\begin{aligned}&\mathcal{F}_{\{\delta_n\}}(u;A)\\&\quad:= \inf\left\{\liminf_{n\to\infty}\frac{1}{\delta_n}\int_A f\bigl(x,v_n(x),\delta_n\nabla v_n(x)\bigr)\,\mathrm{d}x : v_n \to u \text{ in } L^1\bigl(A;\mathbb{R}^d\bigr),\ v_n \in H^1\bigl(A;\mathbb{R}^d\bigr)\right\},\end{aligned}$$

for every $u \in L^1(\Omega;\mathbb{R}^d)$ and $C \in \mathcal{R}$, there exists a sequence $\{u^C_{\varepsilon_n^{\mathcal{R}}}\} \subset H^1(C;\mathbb{R}^d)$ such that

$$u^C_{\varepsilon_n^{\mathcal{R}}} \to u \quad \text{in } L^1\bigl(C;\mathbb{R}^d\bigr)$$

and

$$\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;C) = \lim_{n\to\infty}\frac{1}{\varepsilon_n^{\mathcal{R}}}\int_C f\bigl(x,u^C_{\varepsilon_n^{\mathcal{R}}}(x),\varepsilon_n^{\mathcal{R}}\nabla u^C_{\varepsilon_n^{\mathcal{R}}}(x)\bigr)\,\mathrm{d}x.$$

Claim 1. *$\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)$ is a finite non-negative Radon measure, absolutely continuous with respect to $\mathcal{H}^{N-1}\lfloor\,\partial^* A_0$.*

Claim 2. *The following inequality holds*:

$$\frac{\mathrm{d}\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor\,\partial^* A_0}(x_0) \leqslant K\bigl(x_0,a,b,\nu(x_0)\bigr) \quad \textit{for } \mathcal{H}^{N-1}\textit{-a.e. } x_0 \in \Omega\cap\partial^* A_0.$$

Assuming that Claims 1 and 2 hold, we obtain

$$\begin{aligned}\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) &= \int_\Omega \frac{\mathrm{d}\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor\,\partial^* A_0}(x)\,\mathrm{d}\mathcal{H}^{N-1}\lfloor\,\partial^* A_0(x) = \int_{\Omega\cap\partial^* A_0}\frac{\mathrm{d}\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor\,\partial^* A_0}(x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\\&\leqslant \int_{\Omega\cap\partial^* A_0} K\bigl(x,a,b,\nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x).\end{aligned}$$

In view of Proposition 3.1, we deduce that, in fact,

$$\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) = \int_{\Omega\cap\partial^* A_0} K\bigl(x,a,b,\nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x),$$

and the conclusion in this case follows by a simple diagonalization argument. To finish Step 1 it remains to prove the claims.

Proof of Claim 1. For each $k \in \mathbb{N}$, let $\{v_n^k\} \subset H^1(\Omega;\mathbb{R}^d)$ be such that $\lim_{n\to\infty}\|v_n^k - u\|_{L^1(\Omega;\mathbb{R}^d)} = 0$, and

$$\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) \leqslant \liminf_{n\to\infty}\frac{1}{\varepsilon_n^{\mathcal{R}}}\int_\Omega f\bigl(x,v_n^k(x),\varepsilon_n^{\mathcal{R}}\nabla v_n^k(x)\bigr)\,\mathrm{d}x \leqslant \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) + \frac{1}{k}.$$

Extract $\{n(j,k)\}_j \subset \{n\}$ such that

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n^{\mathcal{R}}} \int_\Omega f\big(x, v_n^k(x), \varepsilon_n^{\mathcal{R}} \nabla v_n^k(x)\big)\,\mathrm{d}x = \lim_{j\to\infty} \frac{1}{\varepsilon_{n(j,k)}^{\mathcal{R}}} \int_\Omega f\big(x, v_{n(j,k)}^k(x), \varepsilon_{n(j,k)}^{\mathcal{R}} \nabla v_{n(j,k)}^k(x)\big)\,\mathrm{d}x.$$

We have

$$\begin{aligned}&\lim_{k\to\infty}\lim_{j\to\infty} \frac{1}{\varepsilon_{n(j,k)}^{\mathcal{R}}} \int_\Omega f\big(x, v_{n(j,k)}^k(x), \varepsilon_{n(j,k)}^{\mathcal{R}} \nabla v_{n(j,k)}^k(x)\big)\,\mathrm{d}x\\ &\quad= \lim_{k\to\infty}\liminf_{n\to\infty} \frac{1}{\varepsilon_n^{\mathcal{R}}} \int_\Omega f\big(x, v_n^k(x), \varepsilon_n^{\mathcal{R}} \nabla v_n^k(x)\big)\,\mathrm{d}x = \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega).\end{aligned}$$

We can extract a subsequence $\{j(k)\} \subset \{j\}$ such that

$$\lim_{k\to\infty} \frac{1}{\varepsilon_{n_k}^{\mathcal{R}}} \int_\Omega f\big(x, v_k(x), \varepsilon_{n_k}^{\mathcal{R}} \nabla v_k(x)\big)\,\mathrm{d}x = \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega),$$

where we have denoted $n_k := n(j(k),k)$ and $v_k := v_{n(j(k),k)}^k$.

The sequence of measures $\{\mu_k\}$, where $\mu_k := \frac{1}{\varepsilon_{n_k}^{\mathcal{R}}} f(x, v_k(x), \varepsilon_{n_k}^{\mathcal{R}} \nabla v_k(x))\mathcal{L}^N \lfloor\, \Omega$, is bounded in $\mathcal{M}(\Omega)$. Thus, there exists a non-negative Radon measure μ such that, up to a subsequence (not relabeled), $\mu_k \rightharpoonup \mu$ weakly* in $\mathcal{M}(\Omega)$. We want to show that $\mu(A) = \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A)$ for all $A \in \mathcal{A}(\Omega)$. To this end, we will verify conditions (i)–(iv) of Lemma 7.3 in [9] (see also [17]), with $\pi : \mathcal{A}(\Omega) \to [0,\infty)$ defined by

$$\pi(A) := \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A).$$

Precisely, for any $A,\ B,\ C \in \mathcal{A}(\Omega)$,

- (i) if $\overline{C} \subset B \subset A$, then $\pi(A) \leqslant \pi(A\backslash\overline{C}) + \pi(B)$,
- (ii) for any $\varepsilon > 0$, there exists $C_\varepsilon \in \mathcal{A}(\Omega)$ with $\overline{C_\varepsilon} \subset A$ and $\pi(A\backslash\overline{C_\varepsilon}) \leqslant \varepsilon$,
- (iii) $\pi(\Omega) \geqslant \mu(\mathbb{R}^N)$,
- (iv) $\pi(A) \leqslant \mu(\overline{A})$.

Note first that (iii) follows immediately from

$$\mu(\mathbb{R}^N) \leqslant \lim_{k\to\infty} \mu_k(\mathbb{R}^N) = \lim_{k\to\infty} \int_\Omega \frac{1}{\varepsilon_{n_k}^{\mathcal{R}}} f\big(x, v_k(x), \varepsilon_{n_k}^{\mathcal{R}} \nabla v_k(x)\big)\,\mathrm{d}x = \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) = \pi(\Omega).$$

Let $\rho : \mathbb{R}^N \to [0,+\infty)$ be a symmetric mollifier, and define

$$\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}}(x) := \frac{1}{(\varepsilon_n^{\mathcal{R}})^N} \rho\bigg(\frac{x}{\varepsilon_n^{\mathcal{R}}}\bigg). \tag{4.6}$$

The sequence $\{\overline{u}_n\} \subset H^1(A;\mathbb{R}^d)$, where $\overline{u}_n := v_k$ if $n = n_k$ for some $k \in \mathbb{N}$, and $\overline{u}_n := u * \rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}}$ if $n \notin \{n_k\colon k \in \mathbb{N}\}$, is admissible for the definition of $\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A)$. We obtain that

$$\begin{aligned}\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A) &\leqslant \liminf_{n\to\infty} \frac{1}{\varepsilon_n^{\mathcal{R}}} \int_A f(x, \overline{u}_n(x), \varepsilon_n^{\mathcal{R}} \nabla \overline{u}_n(x))\,\mathrm{d}x \\ &\leqslant \liminf_{k\to\infty} \frac{1}{\varepsilon_{n_k}^{\mathcal{R}}} \int_A f\big(x, v_k(x), \varepsilon_{n_k}^{\mathcal{R}} \nabla v_k(x)\big)\,\mathrm{d}x \\ &\leqslant \limsup_{k\to\infty} \mu_k(A) \leqslant \mu(\overline{A}),\end{aligned}$$

thus asserting (iv).

To prove (ii), we first show that there exists a constant $C > 0$, such that

$$\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A) \leqslant C\mathcal{H}^{N-1}\big(A \cap \partial^* A_0\big) \quad \text{for all } A \in \mathcal{A}(\Omega). \tag{4.7}$$

Indeed, letting $u_n := \rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u$ ($n \in \mathbb{N}$) we have, by means of the growth conditions in (H2), and since A_0 is polyhedral,

$$\begin{aligned}\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A) &\leqslant \liminf_{n\to\infty} \frac{1}{\varepsilon_n^{\mathcal{R}}} \int_A f\big(x, u_n(x), \varepsilon_n^{\mathcal{R}} \nabla u_n(x)\big)\,\mathrm{d}x \\ &= \liminf_{n\to\infty} \frac{1}{\varepsilon_n^{\mathcal{R}}} \int_{\{x\in A:\ \mathrm{dist}(x,\partial^* A_0)\leqslant \varepsilon_n^{\mathcal{R}}\}} f\big(x, u_n(x), \varepsilon_n^{\mathcal{R}} \nabla u_n(x)\big)\,\mathrm{d}x \\ &\leqslant \liminf_{n\to\infty} \frac{C}{\varepsilon_n^{\mathcal{R}}} \int_{\{x\in A:\ \mathrm{dist}(x,\partial^* A_0)\leqslant \varepsilon_n^{\mathcal{R}}\}} \big(1 + |u_n|^q + \big(\varepsilon_n^{\mathcal{R}}\big)^2 |\nabla u_n|^2\big)\,\mathrm{d}x \\ &\leqslant C \liminf_{n\to\infty} \frac{\mathcal{L}^N(\{x \in A\colon\ \mathrm{dist}(x,\partial^* A_0) \leqslant \varepsilon_n^{\mathcal{R}}\})}{\varepsilon_n^{\mathcal{R}}} = C\mathcal{H}^{N-1}\big(A \cap \partial^* A_0\big).\end{aligned}$$

In view of (4.7), and using the inner regularity of the Radon measure $C\mathcal{H}^{N-1}\lfloor\,\partial^* A_0$, we deduce (ii).

It remains to show that (i) holds. To this aim, let $A,\ B,\ C \in \mathcal{A}(\Omega)$ be such that $\overline{C} \subset B \subset A$. For $\delta > 0$, let B^δ and D^δ be two elements of $\mathcal{R}$ such that $B^\delta \subset B,\ D^\delta \subset A \setminus \overline{C}$, and

$$\mathcal{H}^{N-1}\big((B \setminus B^\delta) \cap \partial^* A_0\big) < \delta, \qquad \mathcal{H}^{N-1}\big(((A \setminus \overline{C}) \setminus D^\delta) \cap \partial^* A_0\big) < \delta. \tag{4.8}$$

Let $\{u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}\}$ and $\{u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\}$ be sequences in $H^1(B^\delta;\mathbb{R}^d)$ and $H^1(D^\delta;\mathbb{R}^d)$, respectively, such that $u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} \to u$ in $L^1(B^\delta;\mathbb{R}^d)$, $u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} \to u$ in $L^1(D^\delta;\mathbb{R}^d)$,

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n^{\mathcal{R}}} \int_{B^\delta} f\big(x, u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}(x), \varepsilon_n^{\mathcal{R}} \nabla u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}(x)\big)\,\mathrm{d}x = \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}\big(u;B^\delta\big) < +\infty, \tag{4.9}$$

and

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n^{\mathcal{R}}} \int_{D^\delta} f\big(x, u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}(x), \varepsilon_n^{\mathcal{R}} \nabla u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}(x)\big)\,\mathrm{d}x = \mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}\big(u;D^\delta\big) < +\infty. \tag{4.10}$$

We may assume, without loss of generality, that

$$u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} = \rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}} * u \quad \text{on } \partial B^\delta, \qquad u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} \to u \quad \text{in } L^q(B^\delta;\mathbb{R}^d) \text{ and } \mathcal{L}^N\text{-a.e. } x \in B^\delta.$$

The proof of this fact follows along the lines of Step 2 of the proof of Lemma 3.2, where we replace Q by B^δ, and v_n by $\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}} * u$ (now $\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}}$ is given by (4.6)). We also note that in this case supp $\nabla(\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}} * u) \subset \{x\colon \operatorname{dist}(x, \partial^* A_0) < \varepsilon^{\mathcal{R}}_n\}$, and for each $k \in \mathbb{N}$, the layer L_k in Step 2 of the proof of Lemma 3.2 should now be taken to be $L_k := \{x \in B\colon \operatorname{dist}(x, \partial B) \leqslant \frac{1}{k}\}$.

Similarly, we may assume that

$$u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n} = \rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}} * u \quad \text{on } \partial D^\delta, \qquad u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n} \to u \quad \text{in } L^q(D^\delta;\mathbb{R}^d) \text{ and } \mathcal{L}^N\text{-a.e. } x \in D^\delta.$$

Extend $u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}$ as $\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}} * u$ outside B^δ, and $u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}$ as $\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}} * u$ outside D^δ. Note that, in view of (3.3),

$$\lim_{n\to\infty} \|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} - u\|_{L^q(A;\mathbb{R}^d)} = \lim_{n\to\infty} \|u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n} - u\|_{L^q(A;\mathbb{R}^d)} = 0. \tag{4.11}$$

Write $B \setminus \overline{C}$ as a union of M_n layers $L^{(i)}_n$ ($i = 1, \ldots, M_n$) of width $\varepsilon^{\mathcal{R}}_n \|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|^{1/2}_{L^2(A;\mathbb{R}^d)}$ so that

$$M_n \cdot \varepsilon^{\mathcal{R}}_n \|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|^{1/2}_{L^2(A;\mathbb{R}^d)} = \mathrm{O}(1). \tag{4.12}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{M_n} \int_{L^{(i)}_n} \biggl(1 + |u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + |u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 + \frac{|u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2}{\|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|_{L^2(A;\mathbb{R}^d)}}\biggr)\,\mathrm{d}x \\
&\quad = \int_{B\setminus\overline{C}} \biggl(1 + |u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + |u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 + \frac{|u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2}{\|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|_{L^2(A;\mathbb{R}^d)}}\biggr)\,\mathrm{d}x,
\end{aligned}$$

and thus there exists $i_0 \in \{1, \ldots, M_n\}$ such that

$$\begin{aligned}
&\int_{L^{(i_0)}_n} \biggl(1 + |u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + |u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 + \frac{|u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2}{\|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|_{L^2(A;\mathbb{R}^d)}}\biggr)\,\mathrm{d}x \\
&\quad \leqslant \frac{1}{M_n} \int_{B\setminus\overline{C}} \biggl(1 + |u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + |u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^q + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 + (\varepsilon^{\mathcal{R}}_n)^2 |\nabla u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2 \\
&\qquad + \frac{|u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2}{\|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n} - u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|_{L^2(A;\mathbb{R}^d)}}\biggr)\,\mathrm{d}x.
\end{aligned} \tag{4.13}$$

We remark that by (4.9), (4.10), (4.11), and (H2),

$$\sup_{n\in\mathbb{N}}\int_{B\setminus\overline{C}}\biggl(1+\bigl|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}\bigr|^q+\bigl|u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\bigr|^q+\bigl(\varepsilon^{\mathcal{R}}_n\bigr)^2\bigl|\nabla u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}\bigr|^2+\bigl(\varepsilon^{\mathcal{R}}_n\bigr)^2\bigl|\nabla u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\bigr|^2+\frac{|u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}-u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}|^2}{\|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}-u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|_{L^2(A;\mathbb{R}^d)}}\biggr)\,\mathrm{d}x$$
$$=: c_0<+\infty. \tag{4.14}$$

Consider cut-off functions $\varphi_n\in C^\infty_c(\mathbb{R}^N;[0,1])$ such that

$$\varphi_n(x)=0 \quad \text{if } x\in\Biggl(\bigcup_{j=i_0+1}^{M_n}L^{(j)}_n\Biggr)\cup\bigl(A\setminus\overline{B}\bigr),$$
$$\varphi_n(x)=1 \quad \text{if } x\in\Biggl(\bigcup_{j=1}^{i_0-1}L^{(j)}_n\Biggr)\cup C,$$

and

$$\|\nabla\varphi_n\|_\infty=\mathrm{O}\biggl(\frac{1}{\varepsilon^{\mathcal{R}}_n\|u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}-u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\|^{1/2}_{L^2(A;\mathbb{R}^d)}}\biggr).$$

Define

$$u_n:=\varphi_n u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}+(1-\varphi_n)u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}.$$

We have that $u_n\to u$ in $L^1(A;\mathbb{R}^d)$ as $n\to\infty$, and in view of (4.9), (4.10),

$$\begin{aligned}
\mathcal{F}_{\{\varepsilon^{\mathcal{R}}_n\}}(u;A) &\leqslant \liminf_{n\to\infty}\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_A f\bigl(x,u_n(x),\varepsilon^{\mathcal{R}}_n\nabla u_n(x)\bigr)\,\mathrm{d}x\\
&\leqslant \liminf_{n\to\infty}\biggl\{\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_{(B\setminus B^\delta)\cup((A\setminus\overline{C})\setminus D^\delta)} f\bigl(x,\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}}*u,\varepsilon^{\mathcal{R}}_n\nabla\bigl(\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}}*u\bigr)\bigr)\,\mathrm{d}x\\
&\quad+\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_{D^\delta} f\bigl(x,u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n},\varepsilon^{\mathcal{R}}_n\nabla u^{D^\delta}_{\varepsilon^{\mathcal{R}}_n}\bigr)\,\mathrm{d}x\\
&\quad+\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_{B^\delta} f\bigl(x,u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n},\varepsilon^{\mathcal{R}}_n\nabla u^{B^\delta}_{\varepsilon^{\mathcal{R}}_n}\bigr)\,\mathrm{d}x+\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_{L^{(i_0)}_n} f\bigl(x,u_n,\varepsilon^{\mathcal{R}}_n\nabla u_n\bigr)\,\mathrm{d}x\biggr\}\\
&\leqslant \limsup_{n\to\infty}\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_{(B\setminus B^\delta)\cup((A\setminus\overline{C})\setminus D^\delta)} f\bigl(x,\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}}*u,\varepsilon^{\mathcal{R}}_n\nabla\bigl(\rho_{\frac{1}{\varepsilon^{\mathcal{R}}_n}}*u\bigr)\bigr)\,\mathrm{d}x+\mathcal{F}_{\{\varepsilon^{\mathcal{R}}_n\}}\bigl(u;D^\delta\bigr)\\
&\quad+\mathcal{F}_{\{\varepsilon^{\mathcal{R}}_n\}}\bigl(u;B^\delta\bigr)+\limsup_{n\to\infty}\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_{L^{(i_0)}_n} f\bigl(x,u_n,\varepsilon^{\mathcal{R}}_n\nabla u_n\bigr)\,\mathrm{d}x\\
&\leqslant \mathcal{H}^{N-1}\bigl((B\setminus B^\delta)\cap\partial^*A_0\bigr)+\mathcal{H}^{N-1}\bigl(((A\setminus\overline{C})\setminus D^\delta)\cap\partial^*A_0\bigr)+\mathcal{F}_{\{\varepsilon^{\mathcal{R}}_n\}}\bigl(u;D^\delta\bigr)\\
&\quad+\mathcal{F}_{\{\varepsilon^{\mathcal{R}}_n\}}\bigl(u;B^\delta\bigr)+\limsup_{n\to\infty}\frac{1}{\varepsilon^{\mathcal{R}}_n}\int_{L^{(i_0)}_n} f\bigl(x,u_n,\varepsilon^{\mathcal{R}}_n\nabla u_n\bigr)\,\mathrm{d}x.
\end{aligned} \tag{4.15}$$

By (4.12), (4.13), and the growth conditions in (H2), we obtain that

$$\begin{aligned}
&\frac{1}{\varepsilon_n^{\mathcal{R}}}\int_{L_n^{(i_0)}} f\bigl(x,u_n(x),\varepsilon_n^{\mathcal{R}}\nabla u_n(x)\bigr)\,\mathrm{d}x\\
&\quad\leqslant \frac{C}{\varepsilon_n^{\mathcal{R}}}\int_{L_n^{(i_0)}}\bigl(1+\bigl|u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^q+\bigl|u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^q+\bigl(\varepsilon_n^{\mathcal{R}}\bigr)^2\bigl|\nabla u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^2+\bigl(\varepsilon_n^{\mathcal{R}}\bigr)^2\bigl|\nabla u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^2\\
&\qquad+\bigl(\varepsilon_n^{\mathcal{R}}\bigr)^2\|\nabla\varphi_n\|_\infty^2\bigl|u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}-u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^2\bigr)\,\mathrm{d}x\\
&\quad\leqslant \frac{C}{\varepsilon_n^{\mathcal{R}}M_n}\int_{B\setminus\overline{C}}\biggl(1+\bigl|u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^q+\bigl|u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^q+\bigl(\varepsilon_n^{\mathcal{R}}\bigr)^2\bigl|\nabla u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^2+\bigl(\varepsilon_n^{\mathcal{R}}\bigr)^2\bigl|\nabla u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^2\\
&\qquad+\frac{\bigl|u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}-u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr|^2}{\bigl\|u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}-u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr\|_{L^2(A;\mathbb{R}^d)}}\biggr)\,\mathrm{d}x\\
&\quad\leqslant c_0C\bigl\|u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}-u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\bigr\|^{1/2}_{L^2(A;\mathbb{R}^d)},
\end{aligned}$$

in view of (4.14). Thus,

$$\limsup_{n\to\infty}\frac{1}{\varepsilon_n^{\mathcal{R}}}\int_{L_n^{(i_0)}} f\bigl(x,u_n(x),\varepsilon_n^{\mathcal{R}}\nabla u_n(x)\bigr)\,\mathrm{d}x=0$$

and we deduce from (4.8) and (4.15) that

$$\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A)\leqslant 2\delta+\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}\bigl(u;B^\delta\bigr)+\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}\bigl(u;D^\delta\bigr)\leqslant 2\delta+\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;B)+\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}\bigl(u;A\setminus\overline{C}\bigr).$$

Letting $\delta\to 0^+$ we obtain that (i) holds. This concludes the proof of Claim 1. □

Proof of Claim 2. Since nearby $\mathcal{H}^{N-1}$-a.e. $x_0\in\partial^*A_0$, u has a planar interface, we can apply Lemma 4.2 to deduce that for $\varepsilon>0$ sufficiently small there exists $\{u_n^{(\varepsilon)}\}\subset H^1(Q_{\nu(x_0)}(x_0,\varepsilon);\mathbb{R}^d)$ such that $u_n^{(\varepsilon)}\to u$ in $L^1(Q_{\nu(x_0)}(x_0,\varepsilon);\mathbb{R}^d)$, and

$$\varepsilon^{N-1}K\bigl(x_0,a,b,\nu(x_0)\bigr)=\lim_{n\to\infty}\frac{1}{\varepsilon_n^{\mathcal{R}}}\int_{Q_{\nu(x_0)}(x_0,\varepsilon)} f\bigl(x_0,u_n^{(\varepsilon)}(x),\varepsilon_n^{\mathcal{R}}\nabla u_n^{(\varepsilon)}(x)\bigr)\,\mathrm{d}x.\tag{4.16}$$

Taking into account (2.2) and (4.16), we obtain

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor\,\partial^*A_0}(x_0)&=\lim_{\varepsilon\to0^+}\frac{\mathcal{F}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;Q_{\nu(x_0)}(x_0,\varepsilon))}{\varepsilon^{N-1}}\\
&\leqslant\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N-1}}\liminf_{n\to\infty}\frac{1}{\varepsilon_n^{\mathcal{R}}}\int_{Q_{\nu(x_0)}(x_0,\varepsilon)} f\bigl(x_0,u_n^{(\varepsilon)}(x),\varepsilon_n^{\mathcal{R}}\nabla u_n^{(\varepsilon)}(x)\bigr)\,\mathrm{d}x\\
&=K\bigl(x_0,a,b,\nu(x_0)\bigr).\qquad\square
\end{aligned}$$

Step 2. We are now ready to consider the general case. In view of Theorem 2.3, there exist polyhedral sets A_k such that $\chi_{A_k}\to\chi_{A_0}$ in $L^1(\Omega)$, $\mathrm{Per}_\Omega(A_k)\to\mathrm{Per}_\Omega(A_0)$, $\mathcal{L}^N(A_k)=\mathcal{L}^N(A_0)$ and

$\mathcal{H}^{N-1}(\partial^* A_k \cap \partial\Omega) = 0$. By Step 1, for every $k \in \mathbb{N}$, there exist sequences $u_n^{(k)} \to \chi_{A_k} a + (1 - \chi_{A_k}) b$ as $n \to \infty$ in $L^1(\Omega; \mathbb{R}^d)$, such that

$$\lim_{n\to\infty} \int_\Omega \frac{1}{\varepsilon_n^{\mathcal{R}}} f\bigl(x, u_n^{(k)}(x), \varepsilon_n^{\mathcal{R}} \nabla u_n^{(k)}(x)\bigr)\,\mathrm{d}x = \int_{\partial^* A_k \cap \Omega} K\bigl(x, a, b, \nu_k(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x),$$

where $\nu_k(x)$ is the measure theoretic unit normal to $\partial^* A_k$ at x. Clearly,

$$\lim_{k\to\infty} \lim_{n\to\infty} \bigl\|u_n^{(k)} - u\bigr\|_{L^1(\Omega;\mathbb{R}^d)} = 0,$$

and for every continuous function $h : \Omega \times \mathbb{R}^N \to [0, +\infty)$, we have (see [16,21,27])

$$\lim_{k\to\infty} \int_{\partial^* A_k \cap \Omega} h\bigl(x, \nu_k(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x) = \int_{\partial^* A_0 \cap \Omega} h\bigl(x, \nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

As $K(\cdot, a, b, \cdot)$ is upper semicontinuous (see Proposition 4.3(ii)), there exist continuous functions $h_m : \Omega \times \mathbb{R}^N \to [0, +\infty)$ such that

$$K(x, a, b, \xi) \leqslant h_m(x, \xi) \leqslant C|\xi|$$

and

$$K(x, a, b, \xi) = \inf_m h_m(x, \xi)$$

for every $(x, \xi) \in \Omega \times \mathbb{R}^N$, where we have extended $K(x, a, b, \cdot)$ as a homogeneous function of degree one (see [18]). Thus, for all $m \in \mathbb{N}$,

$$\begin{aligned}\limsup_{k\to\infty} \lim_{n\to\infty} \int_\Omega \frac{1}{\varepsilon_n^{\mathcal{R}}} f\bigl(x, u_n^{(k)}(x), \varepsilon_n^{\mathcal{R}} \nabla u_n^{(k)}(x)\bigr)\,\mathrm{d}x &= \limsup_{k\to\infty} \int_{\partial^* A_k \cap \Omega} K\bigl(x, a, b, \nu_k(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x) \\ &\leqslant \limsup_{k\to\infty} \int_{\partial^* A_k \cap \Omega} h_m\bigl(x, \nu_k(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x) \\ &= \int_{\partial^* A_0 \cap \Omega} h_m\bigl(x, \nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x).\end{aligned}$$

Taking the limit as $m \to +\infty$ and using Lebesgue's Monotone Convergence Theorem, we deduce that

$$\limsup_{k\to\infty} \lim_{n\to\infty} \int_\Omega \frac{1}{\varepsilon_n^{\mathcal{R}}} f\bigl(x, u_n^{(k)}(x), \varepsilon_n^{\mathcal{R}} \nabla u_n^{(k)}(x)\bigr)\,\mathrm{d}x \leqslant \int_{\partial^* A_0 \cap \Omega} K\bigl(x, a, b, \nu(x)\bigr)\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

In view of Proposition 3.1, and by means of a standard diagonalization procedure, we conclude the proof. □

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