EXPECTED MEAN SQUARES

So far in class we have assumed that treatments are always a fixed effect.

If some or all factors in an experiment are considered random effects, we need to be concerned about the denominator of the F-test because it may not be the Error MS.

To determine the appropriate denominator of the F-test, we need to know how to write the Expected Mean Squares for all sources of variation.

<u>All Random Model</u>

Each source of variation will consist of a linear combination of σ^2 plus variance components whose subscript matches at least one letter in the source of variation.

The coefficients for the identified variance components will be the letters not found in the subscript of the variance components.

<u>Example – RCBD with a 3x4 Factorial Arrangement</u>

Sources of variation	σ^2	$r\sigma^2_{AB}$	$ra\sigma^2_B$	$rb\sigma^2_A$	$ab\sigma^2_R$
Rep	$\sigma^2 + ab\sigma^2_R$				
A	$\sigma^2 + r\sigma^2_{AB} + rb\sigma^2_A$				
B	$\sigma^2 + r\sigma^2_{AB} + ra\sigma^2_B$				
AxB	$\sigma^2 + r\sigma^2_{AB}$				
Error	σ^2				

Step 1. Write the list of variance components across the top of the table.

- There will be one variance component for each source of variation except Total.
- The subscript for each variance component will correspond to each source of variation.
- The variance component for error receives no subscript.

Sources of variation	σ^2	σ^2_{AB}	σ^2_B	σ^2_A	σ^2_R
Rep					
A					
B					
AxB					
Error					

Step 2. Write in the coefficients for each variance component.

- Remember that the coefficient is the letter(s) missing in the subscript.
- The coefficient for Error is the number 1.

Sources of variation	σ^2	$r\sigma^2_{AB}$	$ra\sigma^2_B$	$rb\sigma^2_A$	$ab\sigma^2_R$
Rep					
A					
B					
AxB					
Error					

Step 3. All sources of variation will have σ^2 (i.e. the expected mean square for error as a variance component).

Sources of variation	σ^2	$r\sigma^2_{AB}$	$ra\sigma^2_B$	$rb\sigma^2_A$	$ab\sigma^2_R$
Rep	σ^2				
A	σ^2				
B	σ^2				
AxB	σ^2				
Error	σ^2				

Step 4. The remaining variance components will be those whose subscript matches at least one letter in the corresponding source of variation.

SOV	σ^2	$r\sigma^2_{AB}$	$ra\sigma^2_B$	$rb\sigma^2_A$	$ab\sigma^2_R$
Rep	$\sigma^2 + ab\sigma^2_R$		(Those variance components that have at least the letter R)		
A	$\sigma^2 + r\sigma^2_{AB} + rb\sigma^2_A$		(Those variance components that have at least the letter A)		
B	$\sigma^2 + r\sigma^2_{AB} + ra\sigma^2_B$		(Those variance components that have at least the letter B)		
AxB	$\sigma^2 + r\sigma^2_{AB}$		(Those variance components that have at least the letters A and B)		
Error	σ^2				

Example – CRD with a 4x3x2 Factorial Arrangement

Sources of variation	σ^2 $r\sigma^2_{ABC}$ $ra\sigma^2_{BC}$ $rb\sigma^2_{AC}$ $rc\sigma^2_{AB}$ $rab\sigma^2_{C}$ $rac\sigma^2_{B}$ $rbc\sigma^2_{A}$
A	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC} + rc\sigma^2_{AB} + rbc\sigma^2_{A}$
B	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rc\sigma^2_{AB} + rac\sigma^2_{B}$
C	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rb\sigma^2_{AC} + rab\sigma^2_{C}$
AxB	$\sigma^2 + r\sigma^2_{ABC} + rc\sigma^2_{AB}$
AxC	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC}$
BxC	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC}$
AxBxC	$\sigma^2 + r\sigma^2_{ABC}$
Error	σ^2

Step 1. Write the list of variance components across the top of the table.

- There will be one variance component for each source of variation except Total.
- The subscript for each variance component will correspond to each source of variation.
- The variance component for error receives no subscript.

Sources of variation	σ^2	σ^2_{ABC}	σ^2_{BC}	σ^2_{AC}	σ^2_{AB}	σ^2_{C}	σ^2_{B}	σ^2_{A}
A								
B								
C								
AxB								
AxC								
BxC								
AxBxC								
Error								

Step 2. Write in the coefficients for each variance component.

- Remember that the coefficient is the letter(s) missing in the subscript.
- The coefficient for Error is the number 1.

Sources of variation	σ^2	$r\sigma^2_{ABC}$	$ra\sigma^2_{BC}$	$rb\sigma^2_{AC}$	$rc\sigma^2_{AB}$	$rab\sigma^2_{C}$	$rac\sigma^2_{B}$	$rbc\sigma^2_{A}$
A								
B								
C								
AxB								
AxC								
BxC								
AxBxC								
Error								

Step 3. All sources of variation will have σ^2 (i.e. the expected mean square for error as a variance component).

Sources of variation	σ^2	$r\sigma^2_{ABC}$	$ra\sigma^2_{BC}$	$rb\sigma^2_{AC}$	$rc\sigma^2_{AB}$	$rab\sigma^2_{C}$	$rac\sigma^2_{B}$	$rbc\sigma^2_{A}$
A	σ^2							
B	σ^2							
C	σ^2							
AxB	σ^2							
AxC	σ^2							
BxC	σ^2							
AxBxC	σ^2							
Error	σ^2							

Step 4. The remaining variance components will be those whose subscript matches at least one letter in the corresponding source of variation.

SOV	σ^2 $r\sigma^2_{ABC}$ $ra\sigma^2_{BC}$ $rb\sigma^2_{AC}$ $rc\sigma^2_{AB}$ $rab\sigma^2_{C}$ $rac\sigma^2_{B}$ $rbc\sigma^2_{A}$	
A	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC} + rc\sigma^2_{AB} + rbc\sigma^2_{A}$	(Those variance components that have at least the letters A)
B	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rc\sigma^2_{AB} + rac\sigma^2_{B}$	(Those variance components that have at least the letter B)
C	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rb\sigma^2_{AC} + rab\sigma^2_{C}$	(Those variance components that have at least the letter C)
AxB	$\sigma^2 + r\sigma^2_{ABC} + rc\sigma^2_{AB}$	(Those variance components that have at least the letters A and B)
AxC	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC}$	(Those variance components that have at least the letters A and C)
BxC	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC}$	(Those variance components that have at least the letters B and C)
AxBxC	$\sigma^2 + r\sigma^2_{ABC}$	(Those variance components that have at least the letters A, B and C)
Error	σ^2	

All Fixed Effect Model

Step 1. Begin by writing the expected mean squares for an all random model.

Step 2. All but the first and last components will drop out for each source of variation.

Step 3. Rewrite the last term for each source of variation to reflect the fact that the factor is a fixed effect.

Example RCBD with 3x2 Factorial

SOV	Before		After
Rep	$\sigma^2 + ab\sigma_R^2$	⟶	$\sigma^2 + ab\sigma_R^2$
A	$\sigma^2 + r\sigma_{AB}^2 + rb\sigma_A^2$	⟶	$\sigma^2 + rb\frac{\sum \alpha_i^2}{(a-1)}$
B	$\sigma^2 + r\sigma_{AB}^2 + ra\sigma_B^2$	⟶	$\sigma^2 + ra\frac{\sum \beta_j^2}{(b-1)}$
AxB	$\sigma^2 + r\sigma_{AB}^2$	⟶	$\sigma^2 + r\frac{\sum (\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
Error	σ^2	⟶	σ^2

Rules for Writing Fixed Effect Component

Step 1. Coefficients don't change.

Step 2. Replace σ^2 with $\sum$

Step 3. The subscript of the variance component becomes the numerator of the effect.

Step 4. The denominator of the effect is the degrees of freedom.

Example 2 CRD with a Factorial Arrangement

SOV	Before		After
A	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC} + rc\sigma^2_{AB} + rbc\sigma^2_{A}$	$\longrightarrow$	$\sigma^2 + rbc\frac{\sum \alpha_i^2}{(a-1)}$
B	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rc\sigma^2_{AB} + rac\sigma^2_{B}$	$\longrightarrow$	$\sigma^2 + rac\frac{\sum \beta_j^2}{(b-1)}$
C	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rb\sigma^2_{AC} + rab\sigma^2_{C}$	$\longrightarrow$	$\sigma^2 + rab\frac{\sum \gamma_k^2}{(c-1)}$
AxB	$\sigma^2 + r\sigma^2_{ABC} + rc\sigma^2_{AB}$	$\longrightarrow$	$\sigma^2 + rc\frac{\sum (\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
AxC	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC}$	$\longrightarrow$	$\sigma^2 + rb\frac{\sum (\alpha\gamma)_{ik}^2}{(a-1)(c-1)}$
BxC	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC}$	$\longrightarrow$	$\sigma^2 + ra\frac{\sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
AxBxC	$\sigma^2 + r\sigma^2_{ABC}$	$\longrightarrow$	$\sigma^2 + r\frac{\sum (\alpha\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$
Error	σ^2	$\longrightarrow$	σ^2

Mixed Models

For the expected mean squares for all random models, all variance components remained.

For fixed effect models, all components but the first and last are eliminated.

For mixed effect models:

1. The first and last components will remain.

2. Of the remaining components, some will be eliminated based on the following rules:

   a. Always ignore the first and last variance components.

   b. For the remaining variance components, any letter(s) in the subscript used in naming the effect is ignored.

   c. If any remaining letter(s) in the subscript corresponds to a fixed effect, that variance component drops out.

Example 1 – RCBD with a Factorial Arrangement (A fixed and B random)

SOV	Before		After
Rep	$\sigma^2 + ab\sigma_R^2$	→	$\sigma^2 + ab\sigma_R^2$
A	$\sigma^2 + r\sigma_{AB}^2 + rb\sigma_A^2$	→	$\sigma^2 + r\sigma_{AB}^2 + rb\frac{\sum \alpha_i^2}{(a-1)}$
B	$\sigma^2 + r\sigma_{AB}^2 + ra\sigma_B^2$	→	$\sigma^2 + ra\sigma_B^2$
AxB	$\sigma^2 + r\sigma_{AB}^2$	→	$\sigma^2 + r\sigma_{AB}^2$
Error	σ^2	→	σ^2

Steps for each Source of Variation

Error - No change for Error.

AxB - No change for AxB since only the first and last variance components are present.

B - For the middle variance component, cover up the subscript for B, only A is present. Since A is a fixed effect this variance component drops out.

A - For the middle variance component, cover up the subscript for A, only B is present. Since B is a random effect this variance component remains.

Rep - Replicate is always a random effect, so this expected mean square remains the same.

Example 2 CRD with a Factorial Arrangement (A fixed, B and C random)

SOV	Before		After
A	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC} + rc\sigma^2_{AB} + rbc\sigma^2_A$	$\longrightarrow$	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC} + rc\sigma^2_{AB} + rbc\frac{\sum \alpha_i^2}{(a-1)}$
B	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rc\sigma^2_{AB} + rac\sigma^2_B$	$\longrightarrow$	$\sigma^2 + ra\sigma^2_{BC} + rac\sigma^2_B$
C	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC} + rb\sigma^2_{AC} + rab\sigma^2_C$	$\longrightarrow$	$\sigma^2 + ra\sigma^2_{BC} + rab\sigma^2_C$
AxB	$\sigma^2 + r\sigma^2_{ABC} + rc\sigma^2_{AB}$	$\longrightarrow$	$\sigma^2 + r\sigma^2_{ABC} + rc\sigma^2_{AB}$
AxC	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC}$	$\longrightarrow$	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC}$
BxC	$\sigma^2 + r\sigma^2_{ABC} + ra\sigma^2_{BC}$	$\longrightarrow$	$\sigma^2 + ra\sigma^2_{BC}$
AxBxC	$\sigma^2 + r\sigma^2_{ABC}$	$\longrightarrow$	$\sigma^2 + r\sigma^2_{ABC}$
Error	σ^2	$\longrightarrow$	σ^2

Steps for Each Source of Variation

Error - Error remains the same.

AxBxC - The error mean square for AxBxC remains the same since there are only first and last terms.

BxC - Cover up the B and C in the subscript, A remains and corresponds to a fixed effect. Therefore the term drops out.

AxC - Cover up the A and C in the subscript, B remains and corresponds to a random effect. Therefore the term remains.

AxB - Cover up the A and B in the subscript, C remains and corresponds to a random effect. Therefore the term remains.

C - ABC term - Cover up the C term in the subscript, A and B remain. A corresponds to a fixed effect and B corresponds to a random effect. Since one of the terms corresponds to a fixed effect, the variance component drops out.

BC term - Cover up the C term in the subscript, B remains. B corresponds to a random effect. Since B is a random effect, the variance component remains.

AC term - Cover up the C term in the subscript, A remains. A corresponds to a fixed effect. Since A is a fixed effect, the variance component drops out.

B - ABC term - Cover up the B term in the subscript, A and C remain. A corresponds to a fixed effect and C corresponds to a random effect. Since one of the terms corresponds to a fixed effect, the variance component drops out.

BC term - Cover up the B term in the subscript, C remains. C corresponds to a random effect. Since B is a random effect, the variance component remains.

AB term - Cover up the B term in the subscript, A remains. A corresponds to a fixed effect. Since A is a fixed effect, the variance component drops out.

A - ABC term - Cover up the A term in the subscript, B and C remain. B and C correspond to a random effect. Since none of the terms correspond to a fixed effect, the variance component remains.

AC term - Cover up the A term in the subscript, C remains. C corresponds to a random effect. Since C is a random effect, the variance component remains.

AB term - Cover up the A term in the subscript, B remains. B corresponds to a random effect. Since B is a random effect, the variance component remains.

Deciding What to Use as the Denominator of Your F-test

For an all fixed model the Error MS is the denominator of all F-tests.

For an all random or mix model,

1. Ignore the last component of the expected mean square.
2. Look for the expected mean square that now looks this expected mean square.
3. The mean square associated with this expected mean square will be the denominator of the F-test.
4. If you can't find an expected mean square that matches the one mentioned above, then you need to develop a Synthetic Error Term.

Example 1 – RCBD with a Factorial Arrangement (A fixed and B random)

SOV	Expected mean square	MS	F-test
Rep	$\sigma^2 + ab\sigma_R^2$	1	F = MS 1/MS 5
A	$\sigma^2 + r\sigma_{AB}^2 + rb\frac{\sum \alpha_i^2}{(a-1)}$	2	F = MS 2/MS 4
B	$\sigma^2 + ra\sigma_B^2$	3	F = MS 3/MS 5
AxB	$\sigma^2 + r\sigma_{AB}^2$	4	F = MS 4/MS 5
Error	σ^2	5	

Steps for F-tests

F_{AB} - Ignore $r\sigma^2_{AB}$. The expected mean square now looks like the expected mean square for Error. Therefore, the denominator of the F-test is the Error MS.

F_B - Ignore $ra\sigma^2_B$. The expected mean square now looks like the expected mean square for Error. Therefore, the denominator of the F-test is the Error MS.

F_A - Ignore $rb\frac{\sum\alpha_i^2}{(a-1)}$. The expected mean square now looks like the expected mean square for AxB. Therefore, the denominator of the F-test is the AxB MS.

Example 2 CRD with a Factorial Arrangement (A fixed, B and C random)

SOV	Expected Mean Square	MS	F-test
A	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC} + rc\sigma^2_{AB} + rbc\frac{\sum\alpha_i^2}{(a-1)}$	1	(MS 1 + MS 7)/(MS 4 + MS 5)
B	$\sigma^2 + ra\sigma^2_{BC} + rac\sigma^2_B$	2	MS 2/MS 6
C	$\sigma^2 + ra\sigma^2_{BC} + rab\sigma^2_C$	3	MS 3/MS 6
AxB	$\sigma^2 + r\sigma^2_{ABC} + rc\sigma^2_{AB}$	4	MS 4/MS 7
AxC	$\sigma^2 + r\sigma^2_{ABC} + rb\sigma^2_{AC}$	5	MS 5/MS 7
BxC	$\sigma^2 + ra\sigma^2_{BC}$	6	MS 6/MS 8
AxBxC	$\sigma^2 + r\sigma^2_{ABC}$	7	MS 7/MS 8
Error	σ^2	8	

Steps for F-tests

F_{ABC} - Ignore $r\sigma^2_{ABC}$. The expected mean square now looks like the expected mean square for Error. Therefore, the denominator of the F-test is the Error MS.

F_{BC} - Ignore $ra\sigma^2_{BC}$. The expected mean square now looks like the expected mean square for Error. Therefore, the denominator of the F-test is the Error MS.

F_{AC} - Ignore $rb\sigma^2_{AC}$. The expected mean square now looks like the expected mean square for AxBxC. Therefore, the denominator of the F-test is the AxBxC MS.

F_{AB} - Ignore $rcb\sigma^2_{AB}$. The expected mean square now looks like the expected mean square for AxBxC. Therefore, the denominator of the F-test is the AxBxC MS.

F_C - Ignore $rab\sigma^2_C$. The expected mean square now looks like the expected mean square for BxC. Therefore, the denominator of the F-test is the BxC MS.

F_B - Ignore $rac\sigma_B^2$. The expected mean square now looks like the expected mean square for BxC. Therefore, the denominator of the F-test is the BxC MS.

F_A - Ignore $rbc\frac{\sum \alpha_i^2}{(a-1)}$. The expected mean square now looks like none of the expected mean squares. Therefore, we need to develop a <u>Synthetic Mean Square</u>

Need an Expected Mean Square that looks like: $\sigma^2 + r\sigma_{ABC}^2 + rb\sigma_{AC}^2 + rc\sigma_{AB}^2$.

AC = $\sigma^2 + r\sigma_{ABC}^2 + rb\sigma_{AC}^2$ (missing $rc\sigma_{AB}^2$)
and

AB = $\sigma^2 + r\sigma_{ABC}^2 + rc\sigma_{AB}^2$ (missing $rb\sigma_{AC}^2$)

An expected mean square can be found that includes all needed variance components if you sum the expected mean squares of AC and AB.

$$\text{AC MS + AB MS} = 2\sigma^2 + 2r\sigma_{ABC}^2 + rb\sigma_{AC}^2 + rc\sigma_{AB}^2$$

The problem with this sum is that it is too large by $\sigma^2 + r\sigma_{ABC}^2$.

One method would be to get the needed expected mean square is by:

AC MS + AB MS – ABC MS

Thus F_A could be: $\frac{\text{A MS}}{\text{AC MS} + \text{AB MS} - \text{ABC MS}}$

However, this is not the preferred formula for this F-test.

The most appropriate F-test is one in which the number of MS used in the numerator and denominator are similar.

This allows for more accurate estimates of the degrees of freedom associate with the numerator and denominator.

The formula above has one mean square in the numerator and three in the denominator.

The formula for F_A that is most appropriate is

$$\frac{\text{A MS} + \text{ABC MS}}{\text{AC MS} + \text{AB MS}}$$

The numerator and the denominator then become: $2\sigma^2 + 2r\sigma^2_{ABC} + rb\sigma^2_{AC} + rc\sigma^2_{AB}$.

<u>Calculation of Estimated Degrees of Freedom</u>

Calculation of degrees of freedom for the numerator and denominator of the F-test <u>cannot</u> be calculated by adding together the degrees of freedom for the associated mean squares.

For the F-test: $F_A = \frac{\text{A MS} + \text{ABC MS}}{\text{AC MS} + \text{AB MS}}$

The numerator degrees of freedom = $\frac{(\text{A MS} + \text{ABC MS})^2}{\left[\frac{(\text{A MS})^2}{\text{A df}} + \frac{(\text{ABC MS})^2}{\text{ABC df}}\right]}$

The denominator degrees of freedom = $\frac{(\text{AC MS} + \text{AB MS})^2}{\left[\frac{(\text{AC MS})^2}{\text{AC df}} + \frac{(\text{AB MS})^2}{\text{AB df}}\right]}$

<u>Calculation of LSD Values – CRD with a Factorial Arrangement (A fixed, B and C Random)</u>

LSD_{ABC} (0.05) = $t_{0.05/2;\,\text{Error df}}\sqrt{\frac{2\text{Error MS}}{r}}$

LSD_{BC} (0.05) = $t_{0.05/2;\,\text{Error df}}\sqrt{\frac{2\text{Error MS}}{ra}}$

LSD_{AC} (0.05) = $t_{0.05/2;\,\text{ABC df}}\sqrt{\frac{2(\text{ABC MS})}{rb}}$

LSD_{AB} (0.05) = $t_{0.05/2;\,\text{ABC df}}\sqrt{\frac{2(\text{ABC MS})}{rc}}$

$$LSD_C\ (0.05) = t_{0.05/2;\ BC\ df} \sqrt{\frac{2(BC\ MS)}{rab}}$$

$$LSD_B\ (0.05) = t_{0.05/2;\ BC\ df} \sqrt{\frac{2(BC\ MS)}{rac}}$$

$$LSD_A\ (0.05) = t'_{0.05/2;\ Estimated\ df} \sqrt{\frac{2(AC\ MS + AB\ MS - ABC\ MS)}{rbc}}$$

$$\text{Where Estimated df for t' } = \frac{(AC\ MS + AB\ MS - ABC)^2}{\left[\frac{(AC\ MS)^2}{AC\ df} + \frac{(AB\ MS)^2}{AB\ df} + \frac{(ABC\ MS)^2}{ABC\ df}\right]}$$