

# On an amalgamated duplication of a ring

Marco Fontana, Università degli Studi “Roma Tre”

*Preliminary report, joint work with Marco D’Anna (Università di Catania)*

A regular ideal  $I$  of a ring  $R$  is called a *canonical ideal of  $R$*  if each regular fractional ideal  $J$  of  $R$  is  $I$ -reflexive, i.e.  $J = (I : (I : J)) \cong \text{Hom}_R(\text{Hom}_R(J, I), I)$ . Recall that, given a 1-dimensional Cohen-Macaulay ring  $R$ ,  $R$  is a Gorenstein ring if and only if  $R$  has a canonical ideal isomorphic to  $R$ , i.e.  $J = (R : (R : J)) = J_v$  for each regular fractional ideal  $J$  of  $R$  (cf. E. Matlis [5, Chapter XIII] and W. Heinzer, J. Huckaba and I. Papick [3]).

In higher dimension the problem of the relations between Cohen-Macaulay and Gorenstein rings is more delicate. Given a Cohen-Macaulay local ring  $(R, M, k)$  of dimension  $d$ , a *canonical module  $E$*  is an  $R$ -module such that:

$$\dim_k(\text{Ext}_R^i(k, E)) = \delta_{i,d}.$$

It is wellknown that if a Cohen-Macaulay local ring has a canonical module this is uniquely determined, up to isomorphisms. In general, given a Cohen-Macaulay local ring  $R$ ,  $R$  is a Gorenstein ring if and only if  $R$  has a canonical module isomorphic to  $R$  (cf. W. Bruns and J. Herzog [1, Section 3.3]).

If  $(R, M)$  is a Cohen-Macaulay local ring admitting a canonical module  $E$ , then I. Reiten in 1972 proved that the idealization  $R \times E$  (cf. J. Huckaba [4, Section 25]) is a Gorenstein ring. Later, in 1975, R. Fossum, P. Griffith and I. Reiten in proved a more precise statement: If  $(R, M)$  is a Cohen-Macaulay local ring and  $E$  a  $R$ -module, then  $R \times E$  is a Gorenstein ring if and only if the  $R$ -module  $E$  is a canonical module of  $R$ . But, it is easy to see that  $R \times E$  is not a reduced ring, even if  $R$  is an integral domain.

In this talk, I will describe a new general construction, called *the amalgamated duplication of a ring  $R$  along an  $R$ -module  $E$* , which is an ideal in some overring of  $R$  (and so  $E$  is submodule of the total ring of fractions of  $R$ ), and denoted by  $R \bowtie E$ . (Note that, when  $E^2 = 0$ ,  $R \bowtie E$  coincides with the idealization  $R \times E$ .)

M. D'Anna has applied this construction to give an explicit method for constructing a *reduced* Gorenstein local ring associated in a natural way to a Cohen-Macaulay local domain. More precisely: Let  $(R, M)$  be a Cohen-Macaulay local domain. Assume that  $R$  has a canonical module isomorphic to an ideal of  $R$ , denoted by  $\omega_R$ . Then  $R \bowtie \omega_R$  is a Gorenstein local reduced ring, containing  $R$  as a subring, with maximal ideal  $M \bowtie \omega_R = \sqrt{M(R \bowtie \omega_R)}$  and  $\dim(R \bowtie \omega_R) = \dim(R)$ .

After discussing the main properties of the amalgamated duplication  $R \bowtie E$  in relation with pullback-type constructions (cf. M. Fontana [2]), the main result that I will present in this talk is the following theorem that provides a general frame to the previous results concerning the construction of a Gorenstein ring from a given Cohen-Macaulay ring:

*Let  $R$  be a ring admitting a regular ideal  $I$  such that each  $R$ -module is  $I$ -reflexive. Set  $T := R \bowtie I$  and  $I_T := \text{Hom}_R(T, I)$ , then each  $T$ -module is  $I_T$ -reflexive and  $I_T$  is isomorphic as  $T$ -module to  $T$ .*

#### References

1. W. Bruns and J. Herzog, *Cohen-Macaulay rings*. Cambridge University Press, 1998.
2. M. Fontana, *Topologically defined classes of commutative rings*. Ann. Mat. Pura Appl. **123** (1980), 331–355.
3. W. Heinzer, J. Huckaba and I. Papick, *m-canonical ideals in integral domains*. Comm. Algebra **26** (1998), 3021–3043.
4. J. Huckaba, *Commutative rings with zero divisors*. M. Dekker, 1988.
5. E. Matlis, *1-dimensional Cohen-Macaulay rings*. Springer, 1973.